THE NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE ninth summer meeting of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Tuesday and Wednesday, September 2–3, 1902. Morning and afternoon sessions were held on each day in Room 7 of the University Hall. The President of the Society, Professor Eliakim Hastings Moore, presided at the first session, and was relieved at the later sessions by Professor T. S. Fiske and Professor H. S. White.

The attendance numbered about fifty-five, including the following thirty-seven members of the society:

Professor Oskar Bolza, Mr. A. R. Crathorne, Professor L. E. Dickson, Professor L. W. Dowling, Professor John Eiesland, Dr. William Findlay, Professor T. S. Fiske, Dr. W. B. Fite, Dr. J. W. Glover, Professor G. W. Greenwood, Professor A. S. Hathaway, Professor T. F. Holgate, Dr. Edward Kasner, Dr. H. G. Keppel, Professor Kurt Laves, Dr. D. N. Lehmer, Professor Heinrich Maschke, Professor J. A. Miller, Professor E. H. Moore, Professor Frank Morley, Dr. F. R. Moulton, Professor H. B. Newson, Professor Alexander Pell, Professor D. A. Rothrock, Miss I. M. Schottenfels, Professor G. T. Sellew, Professor J. B. Shaw, Professor E. B. Skinner, Professor Ormond Stone, Professor E. J. Townsend, Professor H. W. Tyler, Professor C. A. Waldo, Professor H. S. White, Professor W. H. Williams, Professor J. W. A. Young, Mr. J. W. Young, Professor Alexander Ziwet.

The Council announced the election of the following persons to membership in the Society: Professor T. J. I'a. Bromwich, Queen's College, Galway, Ireland; Mr. J. S. Brown, New York City; Professor G. C. Edwards, University of California, Berkeley, Cal. Eight applications for membership were received.

The Council decided that the name of the new section, authorized at the last meeting, should be the San Francisco Section; and the programme committee of the Section was approved. In accordance with a motion adopted by the Society, the Council appointed a special committee consisting of Professors Tyler (chairman), Fiske, Osgood, Young and Ziwet to

report on standard definitions of requirements in mathematical subjects for admission to colleges and scientific schools. The committee will coöperate with the corresponding committees of the Society for the promotion of engineering education, the National educational association and other interested bodies.

Besides the scientific sessions the members gathered together on various social occasions. In particular may be mentioned a dinner at the old academy building of the University, followed by a visit to the Observatory on invitation from the director, Professor Hough. Before adjourning, the Society by a rising vote expressed to Northwestern University and to the local members of the Society its appreciation of their hospitable efforts in providing for the meeting and the entertainment of the members.

The large attendance and the character of the programme is worthy of comment in view of the fact that this summer meeting is the first one held west of Buffalo.

The following papers were read at this meeting:

(1) Dr. F. R. Moulton: "A method of constructing general expressions for the elements of the planetary orbits which are valid for a finite time."

(2) Professor A. S. Hathaway: "The quaternion treatment of the problem of three bodies."

(3) Dr. J. V. Collins: "A general notation for vector analysis."

(4) Professor L. E. Dickson: "Definitions of a linear associative algebra by independent postulates."

(5) Professor L. E. Dickson: "Definitions of a field by independent postulates."

(6) Dr. E. V. Huntington: "Definitions of a field by sets of independent postulates."

(7) Dr. Otto Dunkel: "Regular singular points of a system of homogeneous linear differential equations of the first order."

(8) Professor Oskar Bolza: "Some instructive examples in the calculus of variations."

(9) Professor J. B. Shaw: "On linear associative algebras."

(10) Dr. W. B. Fite: "Concerning the commutator subgroups of groups whose orders are powers of primes."

(11) Mr. J. W. Young: "On a certain group of linear substitutions and the functions belonging to it" (preliminary report).

(12) PROFESSOR JACOB WESTLUND: "On the class number of the cyclotomic field $k(e^{2\pi i}/p^n)$."

(13) PROFESSOR L. E. DICKSON: "Announcement of new simple groups" (preliminary communication).

(14) PROFESSOR ALFRED LOEWY: "Ueber die Reducibilität der Gruppen linearer homogener Substitutionen."

(15) PROFESSOR H. S. WHITE: "A special twisted cubic with rectilinear directrix."

(16) PROFESSOR F. MORLEY: "Orthocentric properties of the plane n-line."

(17) DR. VIRGIL SNYDER: "Forms of sextic scrolls of genus greater than 1."

(18) PROFESSOR PETER FIELD: "Quintic curves for which $p = 1$."

(19) DR. C. J. KEYSER: "Concerning the line and plane geometries of point four-space, and allied theories."

(20) PROFESSOR ARNOLD EMCH: "A linkage for $u = e^{i\theta}z$."

(21) DR. EDWARD KASNER: "The bilinear point-line connex in space: an extension of Clebsch's connex."

(22) MR. E. A. HOOK: "Multiple points on Lissajous's curves in two and three dimensions."

(23) DR. L. P. EISENHART: "Infinitesimal deformation of the skew helicoid."

(24) PROFESSOR JOHN EIESLAND: "Null systems in space of five dimensions."

(25) PROFESSOR E. D. ROE: "Note on a partial differential equation."

(26) DR. SAUL EPSTEEN: "On integrability by quadratures."

(27) MR. W. B. FORD: "On the possibility of differentiating term by term the developments for an arbitrary function of one real variable in terms of Bessel functions."

(28) PROFESSOR T. F. HOLGATE: "Apolar triads on a straight line and on a conic."

(29) PROFESSOR H. B. NEWSON: "List of continuous groups of collineations in space."

(30) MISS HELEN BREWSTER: "The group of collineations leaving a quadric surface invariant."

(31) PROFESSOR ALEXANDER PELL: "On the generalized Beltrami problem."

(32) PROFESSOR L. HEFFTER: "Ueber Curvenintegrale in m-dimensionalem Raum."

The papers of Dr. Dunkel and Mr. Hook were presented to the Society through Professor Bôcher, those of Professor Loewy and Professor Heffter through Professor Moore, that of Dr. Epsteen through Professor Lovett, and Miss Brewster's through Professor Newson. In the absence of the authors, Dr. Huntington's paper was read by Professor Dickson, Professor Westlund's and Professor Emch's by Professor Moore, Professor Loewy's by Professor Maschke, Dr. Snyder's and Professor Field's by Dr. Fite, and Miss Brewster's by Professor Newson. The papers of Dr. Collins, Dr. Dunkel, Dr. Keyser, Mr. Hook, Dr. Eisenhart, Professor Roe, Mr. Ford, and Professor Heffter were read by title.

Professor Bolza's paper was published in the October number of the BULLETIN. The papers of Dr. Fite and Dr. Epsteen will appear in later numbers of the BULLETIN. Abstracts of the other papers presented are given below. The abstracts are numbered to correspond to the titles in the list above.

1. In constructing expressions for the elements of the planetary orbits three things are desirable: (*a*) that they shall, if they are series, converge for at least a finite interval of time; (*b*) that it shall be possible practically to find their coefficients; and (*c*) that the time for which they are valid shall be very long. The method of developing the elements as power series in the masses can be proved to give results which converge for a finite interval of time, but these series have the practical inconvenience of containing secular terms even of the first order. The thought has been that these secular terms may be the expansions of periodic terms. Following out this idea, Lagrange has succeeded by quite arbitrary and up to the present unjustified processes in avoiding the secular terms of the first order in the masses and in the eccentricities. Secular terms of higher orders appear, and even if they did not, his results in no way prove the stability of the solar system, as is frequently stated. Dr. Moulton inquires whether these secular terms may not be avoided by processes which can be proved to be valid. The answer is in the affirmative, although the method of treatment differs from that of Lagrange. The series which are used fulfill condition (*a*); they fulfill (*b*) about as well as those of the standard methods; and they are probably valid for a much longer time than those now in use.

2. Professor Hathaway employs the quaternion analysis in the problem of three bodies, with the hope that the new and concise forms of that analysis may suggest important developments. Let α, β, γ be the vector sides BC, CA, AB of a fixed triangle of reference, and put

$$S\cdot\beta\gamma = 1/a, \quad S\cdot\gamma\alpha = 1/b, \quad S\cdot\alpha\beta = 1/c, \quad UV\alpha\beta = k.$$

Then if P, Q, R be the positions of masses a, b, c at any time t, there is a definite strain ϕ such that

$$\phi k = 0, \qquad \phi ABC = PQR$$

whose differential equation is

$$(1) \qquad \ddot{\phi} = \phi\left(\frac{bc\alpha S\alpha}{T^3\phi\alpha} + \frac{ca\beta S\beta}{T^3\phi\beta} + \frac{ab\gamma S\gamma}{T^3\phi\gamma}\right) = \phi\pi, \text{ say,}$$

or in conjugate form, $\ddot{\phi}' = \pi\phi'$. The area and energy integrals are

$$(2) \qquad \dot{\phi}\phi' - \phi\dot{\phi}' = V\epsilon, \qquad \tfrac{1}{2}S_1\dot{\phi}'\dot{\phi} + S_1\phi'\phi\pi = C,$$

where ϵ is the perpendicular to the invariant plane, and C is the sum of the kinetic and potential energies.* Forms are reduced when expressed in terms of $\psi = \phi'\dot{\phi}$, and π is already reduced since $T^2\phi\rho = -S\cdot\rho\psi\rho$.

We have $\dot{\phi}'\phi - \phi'\dot{\phi} = xVk$, which gives by differentiation $\pi\psi - \psi\pi = \dot{x}Vk$, a reduced differential equation for x. Also by differentiating ψ twice we find $2\dot{\phi}'\dot{\phi}, = \Phi$, say, in reduced form, so that the identity

$$[k(\dot{\psi} + xVk)k\psi k(\dot{\psi} - xVk) + 4S_2\psi\cdot k\Phi]^2 = 0$$

or

$$(3) \qquad \begin{aligned} x^4 + 2x^2(S\cdot k\psi k\Phi + S_2\dot{\psi}) + 4xS_1\cdot k\psi k\dot{\psi}k\Phi + S_2\dot{\psi}^2 \\ + 4S_2\cdot\psi\Phi - 2S_1\cdot k\dot{\psi}k\psi k\dot{\psi}k\Phi = 0 \end{aligned}$$

is a reduced equation for x itself. This agrees with Lagrange, x being to a constant factor his ρ; but does not agree with

* The cubic in any ϕ is $\phi^3 - S_1\phi\cdot\phi^2 + S_2\phi\cdot\phi - S_3\phi = 0$. Thus S_r is an operator on a strain which gives a characteristic scalar. Products under S_r are cyclically commutative and replaceable by their conjugates. S_1 is distributive over a sum, and S_3 over a product. In the present case $S_3 = 0$, and S_2 has the distributive property of S_3 for coplanar factors.

Dziobek [Planetary motions, page 56 (19)]. I have verified, directly and indirectly, that (3) is a particular integral of the differential equation for x in connection with (1), the general integral being found by replacing x by $x + C$ in (3). This settles a question which Dziobek says was undetermined (Planetary motions, page 57).

By differentiating ψ three times and substituting, we find the reduced form of (1)

$$\dddot{\psi} = 2(\dot{\psi}\pi + \pi\dot{\psi}) + \pi\psi + \psi\pi + x(Vk\pi - \pi Vk). \tag{4}$$

This is equivalent to three ordinary equations of third order between the mutual distances of the bodies, found by operating with S_1, $S_1 \cdot \psi$, $S_1 \cdot \pi$. The first two are straight differentials of the reduced forms of (2), so that with these integrations the final forms are

$$S_1\ddot{\psi} + 2S_1 \cdot \psi\pi = 4C, \tag{5}$$

$$S_1 \cdot \psi\ddot{\psi} = 2S_1 \cdot \psi^2\pi + \tfrac{1}{2}S_1 \cdot \dot{\psi}^2 - x^2 + 2T^2\epsilon, \tag{6}$$

$$S_1 \cdot \pi\dddot{\psi} = 2S_1 \cdot \pi\pi\psi + 4S_1 \cdot \pi^2\dot{\psi}. \tag{7}$$

Dziobek does not note that the area integral (6) is one of the integrals of (4).

3. The different branches of vector analysis are very much alike in several respects. They deal with the same fundamental concept, the vector, and employ largely the same derived concepts, the scalar and vector products and their combinations. Apparently then there is no reason why a single consistent notation should not be used in all the branches of the subject. Instead of this we seem to be tending in the opposite direction, as is evidenced in Professor Gibbs's new book.

Professor Gibbs has made certain improvements in the notation which will doubtless stand. He avoids Hamilton's cumbrous T and U and puts his signs of multiplication *between* instead of to one side of the factors, thus admitting of a new sign of operation being put on either side. This Professor Gibbs was led to do doubtless through his study of dyadics. However, instead of retaining Hamilton's letters or Grassmann's notation for the scalar and vector products, he has introduced two new marks, the dot and the cross, to denote them.

Now all interests can be easily harmonized in the following way: — Let $a^{s}b$, $a^{v}b$, $a^{q}b$, ab denote respectively Sab, Vab, quaternion ab, dyadic ab. Also let $a^{s}b^{v}c$ be denoted by $[abc]$, as is done by Gibbs, whenever this notation is more convenient. In this way are retained Hamilton's notation, and the good points in Gibbs's and Grassmann's notations. There is another very good reason why Gibbs's dot and cross should be given up and the letters retained. It is that the use of letters admits of an easy extension of the notation to new products. That there are such besides the quaternion the author has demonstrated by constructing them and putting them to practical use.

4. In the usual theory of higher complex members, the coördinates are either real numbers or else ordinary complex quantities. To avoid the resulting double phraseology and to attain an evident generalization of the theory, Professor Dickson considers systems of complex numbers whose coördinates belong to an arbitrary field F. The usual definition rests upon a multiplication table for n units. In essence, a system of complex numbers depends upon n^3 constant marks γ_{iks} of F subject to the three conditions

$$\sum_{s=1}^{n} \gamma_{iks}\gamma_{slt} = \sum_{s=1}^{n} \gamma_{kls}\gamma_{ist} \qquad (i, k, l, t = 1, 2, \cdots, n),$$

$$\Delta_a \equiv \left| \sum_{i=1}^{n} \gamma_{iks} a_i \right| \text{ not zero for every } a_1, \cdots, a_n,$$

$$\Delta_a' \equiv \left| \sum_{k=1}^{n} \gamma_{iks} a_k \right| \text{ not zero for every } a_1, \ldots, a_n.$$

These conditions are shown to be independent by using systems with $n = 2$.

For the second definition the elements are $A = (a_1, a_2, \cdots, a_n)$, where $a_1, \cdots, a_n$ are marks of F. Addition is defined thus:

$$A + B = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n).$$

A second rule of combination of elements is defined by four properties: (1) For any two elements A and B of the system, $A \cdot B$ is an element whose coördinates are bilinear functions of the coördinates of A and B, with fixed coefficients belonging to F. (2) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, if $A \cdot B$, etc., belong to the

system. (3) There exists in the system an element I such that $A \cdot I = A$ for every element A. (4) There exists in the system at least one element A such that $A \cdot Z \neq 0$ for any element $Z \neq 0$. The elements are shown to form a system of complex numbers according to the usual definition. The four properties are shown to be independent. The paper will appear in the *Transactions*.

5. Professor Dickson's second paper gives a definition of a field by means of nine independent postulates and a second definition by means of eleven independent postulates. Of various possible sets of postulates, these two sets (especially the second) appear to present most naturally the elementary properties necessarily holding for a field and in a manner most suitable for actually testing for the field property.

A set of elements with two rules of combination denoted by $\circ$ and $\square$ forms a field if the following properties hold: (1) If a and b belong to the set, then also $a \circ b$ belongs to the set. (2) $a \circ b = b \circ a$, whenever $a \circ b$ and $b \circ a$ belong to the set. (3) $(a \circ b) \circ c = a \circ (b \circ c)$, whenever $a \circ b$, $b \circ c$, $(a \circ b) \circ c$, and $a \circ (b \circ c)$ belong to the set. (4) For any two elements a and b of the set, there exists in the set an element x such that $(a \circ x) \circ b = b$. (5), (6), (7) are the same as (1), (2), (3) with $\circ$ replaced by $\square$. (8) For any two elements a and b of the set, such that $c \square a \neq a$ for at least one element c, there exists in the set an element x such that $(a \square x) \square b = b$. (9) $a \square (b \circ c) = (a \square b) \circ (a \square c)$, whenever $a \square b$, $a \square c$, etc., belong to the set.

The second set employs postulates (1), (2), (3), (5), (6), (7), (9) and the following four: (4′) There exists in the set an element z such that $z \circ b = b$ for every element b. (4″) If elements z of character (4′) exist, then for some such element z and for every element a, there exists an element x for which $a \circ x = z$. (8′) There exists an element u such that $u \square b = b$ for every element b. (8″) If elements u of character (8′) exist, then for some such element u and for every element a such that $c \square a \neq a$ for at least one element c, there exists an element x for which $a \square x = u$.

This paper will appear in the *Transactions*.

6. Dr. Huntington's paper presents several sets of postulates which define a field, that is, an assemblage in which all the

rational operations of algebra can be performed. The simplest of these sets contains the following seven postulates, where the symbols $\oplus$ and $\odot$ indicate abstract operations exemplified by addition and multiplication of rational numbers:

1) If a, b and either $a \oplus b$ or $b \oplus a$ belong to the field, then $a \oplus b = b \oplus a$.

2) If a, b, c, $a \oplus b$, $b \oplus c$ and either $(a \oplus b) \oplus c$ or $a \oplus (b \oplus c)$ belong to the field, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

3) For every two elements a and b ($a = b$ or $a \neq b$) there is an element x in the field such that $a \oplus x = b$.

4), 5) The same as 1) and 2), when $\oplus$ is replaced by $\odot$.

6) For every two elements a and b ($a = b$ or $a \neq b$), provided $a \oplus a \neq a$, there is an element y in the field such that $a \odot y = b$.

7) If a, b, c, $b \oplus c$, $a \odot b$, $a \odot c$ and either $a \odot (b \oplus c)$ or $(a \odot b) \oplus (a \odot c)$ belong to the field, then $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

The independence of the postulates of each set is established by the now familiar method of Peano and Hilbert. The paper will appear in the *Transactions*.

7. In this paper, which will be published in the *Proceedings the American Academy of Arts and Sciences*, Mr. Dunkel considers a system of differential equations

$$\frac{dy_i}{dx} = \sum_{j=1}^{j=n} \left(\frac{\mu_{i,j}}{x} + a_{i,j} \right) y_j \qquad (i = 1, 2, \cdots, n)$$

in which the $\mu_{i,j}$'s are constants and the $a_{i,j}$'s are functions of the real independent variable x, not necessarily analytic, but continuous in the interval $0 < x \leqq b$. The absolute values, $|a_{i,j}|$, multiplied by certain positive integral powers of $\log x$ are required to be integrable up to the point $x = 0$; these powers vary according to the set of solutions being developed, and they may in some cases be zero. When the $a_{i,j}$'s satisfy these conditions, the point $x = 0$ is called a regular singular point. Use is made of a theorem on elementary divisors to reduce the system of equations to a canonical form. Solutions of the canonical system are developed by the method of successive approximations; and from these the form of the solutions of the original system is then determined. The results thus obtained are applied to the single differential equation

$$\frac{d^n y}{dx^n} + \left(\frac{\mu_1}{x} + p_1\right)\frac{d^{n-1}y}{dx^{n-1}} + \left(\frac{\mu_2}{x} + p_2\right)\frac{1}{x}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + \left(\frac{\mu_n}{x} + p_n\right)\frac{1}{x^{n-1}}y = 0,$$

in which the μ's are constants, and the p's are functions of the real independent variable x satisfying the same requirements that were imposed upon the functions $a_{i,j}$.

Let r_κ be a root of multiplicity e_κ of the indicial equation. The multiplicities of all the roots of this equation whose real parts are the same as that of r_κ are considered, and e_κ is chosen as any one of these multiplicities as great as any other in this set. It is then required that the absolute values of the p's after being multiplied by $(\log x)^{e_\kappa - 1}$ be integrable up to $x = 0$; and with this requirement it is found that we can develop the following e_κ solutions and their first $n-1$ derivatives corresponding to the root r_κ:

$$y^{\kappa\lambda} = x^{r_\kappa}(\log x)^{e_\kappa - \lambda}E_n^{\kappa\lambda}, \qquad (\lambda = 1, 2, \cdots, e_\kappa),$$

$$\frac{d^i y^{\kappa\lambda}}{dx^i} = x^{r_\kappa - i}(\log x)^{e_\kappa - \lambda}E_{n-i}^{\kappa\lambda},$$

where the functions $E_{n-i}^{\kappa\lambda}$ are continuous in an interval $0 \leqq x \leqq c$, and

$$E_n^{\kappa\lambda}|_{x=0} = 1, \quad E_{n-i}^{\kappa\lambda}|_{x=0} = r_\kappa(r_\kappa - 1)\cdots(r_\kappa - i + 1).$$

In the case $n = 2$ these results not only yield the theorems which were established by Professor Bôcher in the *Transactions* for January, 1900, but for the case of equal exponents go slightly further.

9. Professor Shaw's paper gives the reduction of all numbers of linear associative algebras to typical forms, and a new basis of classification of such algebras. It makes use of a previously discovered form involving units of the type λ_{ijk} such that

$$\lambda_{ijk}\cdot\lambda_{i'j'k'} = 0 \quad \text{if } j \neq i'$$

$$= \lambda_{ij'k+k'} \quad \text{if } j = i'.$$

The units are also subject to restrictions as to the size of the num-

ber k, represented by the third subscript. The method is given of deriving all algebras to which belong numbers of which the most general number satisfies a given characteristic equation. The place of Peirce's, Scheffer's and Molien's investigations is also shown, with an examination of the boundaries of these investigations and an extension and generalization of their results. The paper is a conclusion to one presented before the Chicago Section, January, 1902.

11. The moduli of periodicity for the normal integrals of the first kind on the Riemann surface for $w^4 = (z-\kappa_1)^2(z-\kappa_2)^2(z-\kappa_3)(z-\kappa_4)^3$ depend on a single parameter ω which undergoes a linear substitution by every monodromy of the branch points. Mr. Young's paper considers the group Γ of all such substitutions, which is found to be generated by the two, $\omega' = \omega + \sqrt{2}$ and $\omega' = -1/\omega$. All functions automorphic for the group are shown to be rationally expressible in terms of the anharmonic ratio of the branch points.

The properties of Γ are studied in detail, the investigation following the lines laid down by Klein in his treatment of the (elliptic) modular functions, and results analogous to practically all of Klein's results are obtained. The most noteworthy difference is found in the fact that Γ contains an infinite series of subgroups of genus $p = 0$. These subgroups are of index $2n$ and are isomorphic with the dihedron group of order $2n$; two sets of generators are determined for each.

The paper closes with a discussion of the quotient groups of Γ with the "principal" congruence subgroups, the latter consisting of all substitutions of Γ that are congruent to identity (mod. q). When q is an odd prime, the quotient group is shown to be of order $\frac{1}{2}q(q^2-1)$ or $q(q^2-1)$ according as q is of the form $8h \pm 1$ or $8h \pm 3$. In the former case the quotient group is simply isomorphic with Klein's $G_{\frac{1}{2}q(q^2-1),2}$; in the latter case it contains a subgroup of half its order which is simply isomorphic with $G_{\frac{1}{2}q(q^2-1)}$.

12. Dr. Westlund's paper, which is an extension of a paper presented at the meeting of the Chicago section in January last, treats of the relation between the class numbers of the cyclotomic number field $k\left(e^{\frac{2\pi i}{p^n}}\right)$, when p is any odd prime, and its subfield $k\left(e^{\frac{2\pi i}{p^{n-1}}}\right)$. The method used is similar to that used by Weber for the case $p = 2$.

13. Professor Dickson's third paper is an addition to his article, "Theory of linear groups in an arbitrary field," *Transactions*, volume 2 (1901), pages 363–394. By certain modifications, the methods of §§ 9–10 may be made to apply to the there excluded case of modulus 2, provided $n > 1$. Hence there results a new system of simple groups of order $2^{n^2}(2^{2n}-1)(2^{2n}-1)$, for any integer $n > 1$. There remains for subsequent investigation the case $n = 1$, the order being then 12096. For $n = 2$ and $n = 3$ the orders of the new simple groups are respectively $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$, $2^{18} \cdot 3^5 \cdot 7^2 \cdot 19 \cdot 73$.

14. The paper of Professor Loewy concerns the reducibility of groups of linear homogeneous substitutions, and is in abstract as follows: A group G of linear homogeneous substitutions in n variables is styled reducible if there exist $m < n$ linear homogeneous functions of the variables with constant coefficients, such that under the group they are transformed in a linear homogeneous way. The paper considers the following fundamental theorem: To every reducible group G of linear homogeneous substitutions there belongs a definite finite number of irreducible groups of linear homogeneous substitutions, which are uniquely determined apart from the order, in case one considers similar groups as identical. This theorem is applied in the first place to the theory of groups of a finite number of linear homogeneous substitutions and then to the theory of those infinite groups introduced by the author in volume 53, page 225, of the *Mathematische Annalen* and designated as groups of the type of a finite group. Further, there is introduced the more general notion of reducibility or irreducibility of a group G of linear homogeneous substitutions with respect to a realm of rationality or field (Körper) Ω to which all the coefficients of the various substitutions of the group belong, and the fundamental theorem is treated from this standpoint. In conclusion, the following theorem is proved: In order to exhibit all the finite groups belonging to a given field Ω, it is sufficient to exhibit all the finite groups which are irreducible with respect to the field Ω.

15. A twisted cubic has three osculating planes through every point. There are always two points in which these three planes are perpendicular to one another. If there are more than two such points, there must be an infinite number,

and their locus is then called the directrix of such a specialized cubic curve. For the case where the curve is a parabola, *i. e.*, if the plane at infinity osculates the curve, O. Böklen and W. F. Meyer have shown that its directrix is a straight line, and have discussed the conditions necessary for its existence. Professor White's paper is an examination of the same problem for cubical ellipses and hyperbolas. By analogy with plane conics one would look for a curvilinear directrix, if any; but it is proven that it must be straight. The conditions for its existence are found in the identical vanishing of a ternary quadric, giving three conditions of which, however, only two are independent. Meyer's twisted cubics with a directrix are defined by three conditions, while this new class is subject to only two; the latter, however, does not include the former class, but the two together comprise all possible twisted cubics with directrix.

16. Professor Morley's memoir is a continuation of one published in the *Transactions*, volume 1, page 97. What corresponds, for a plane n-line, to the elementary facts that for a 3-line the perpendiculars meet at a point, and for a 4-line the 4 such points lie on a line?

First, four lines determine uniquely a hypocycloid of class 3, Δ^3; if from the center of the Δ^3 touching each 4 of 5 lines a perpendicular be drawn to the fifth line, these five perpendiculars meet at a point; for 6 lines the 6 such points lie on a line. So there is a quite simple curve Δ^{2n-1} uniquely determined by $2n$ lines; if from the center of the Δ^{2n-1} touching each $2n$ of $2n+1$ lines a perpendicular be drawn to the line omitted, these perpendiculars meet at a point and for $2n+2$ lines the $2n+2$ such points lie on a line. The curve Δ is for metrical purposes very convenient, much more so than Clifford's n-fold parabola, also uniquely determined by $2n$ lines. Second, if from the nine-point center, or center of Feuerbach's circle, of each 3 of 4 lines, a perpendicular be drawn on the line left out, these perpendiculars meet at a point. And the center of a circle arising from n lines has the analogous property. Thus we have for an even number of lines an orthocenter and a directrix; for an odd number, two and therefore a line of orthocenters.

17. Dr. Snyder has applied the same methods which he employed in the study of unicursal and elliptic scrolls of order six to obtain the forms and equations of those of genus greater

than 1. There are eleven types of genus 2, five of genus 3 and two of genus 4. The scrolls cannot have a triple conic nor more than one double conic as part of the nodal curve. Added to the paper is a scheme showing that the maximum genus of a scroll of order $2n$ is $(n-1)^2$, and of a scroll of order $2n-1$ is $n(n-1)$.

18. Professor Field obtains the general equation of a quintic curve having five fixed double points. It follows from the form of this equation that if a conic and cubic be taken and breaks made at two of their points of intersection, there result quintic curves. The curves are classified according to the sequence of the double points. Forty-six forms are obtained.

19. Dr. Keyser's paper is devoted primarily to a systematic introduction to the line and plane geometries of four-space, and secondarily to a correlation of these with kindred doctrines, such as the circle geometry of ordinary space and the theory of the four-space point conceived as a plenum of circular cones. The point and lineoid (three-space) theories of four-space, in so far as needed, are assumed. By reason of the locus and envelope phases of the line and plane, one has to do with four fundamental element aspects. These are strictly four distinct elements, giving rise to four distinct geometric theories which, like the elements, fall into reciprocal pairs. In a first chapter, by means of an arbitrary finite pentaedron of reference, *i. e.*, the configuration determined by any five non-collineoidal points or any five non-copunctal lineoids, four appropriate homogeneous coördinate systems are established, and their relationships disclosed in connection with the homographic and dualistic transformations. Each of these systems is composed of ten variables whose nine ratios, being connected by five (equivalent in general to three independent) quadratic identities, reduce to the necessary and sufficient number six of independent functions. These identities, being essentially like those which appear in the circle theory of ordinary space, are of order five, whence it appears that four arbitrary lines (planes) determine uniquely a fifth and so give rise to a configuration of lines (planes) analogous to the "pentacycle" of Stephanos. A second chapter which is concerned with the solution of preliminary elementary problems in terms of the adopted coördinate systems, is followed by a third devoted to a somewhat detailed exposition of

the above-mentioned correlation. The sphere (of ordinary space) may be conceived as an orthosphere, *i. e.*, a sphere regarded as the assemblage of spheres orthogonal to it; or it may be regarded as the envelope of all the orthospheres containing it. Calling the sphere an orthosphere or a sphere according as it is conceived in the former or in the latter way, and pairing the spheres and orthospheres (of ordinary space) respectively with the points and lineoids of four-space, it is seen that to the union of point and lineoid corresponds orthogonality of sphere and orthosphere — a principle which facilitates the parallelization of a theory of either space with its correspondent in the other. Among such readily deducible correspondences may be mentioned: the circle (and its geometry) has four aspects matching the four element aspects (and corresponding theory phases) mentioned above; the geometry of circles orthogonal to a sphere is analytically identical with the line (plane) geometry of a lineoid (four-space point); the theory of circles in involution with a given circle is parallel to the theory of lines (planes) having a point (or lineoid) in common with a given plane (line); and the geometry of circles in bi-involution with a given circle corresponds proposition for proposition with the four-space geometry of coplanar lines or of collinear planes.

20. In a previous paper, read before the society (Chicago) June 1, 1902, and to be published in the *Transactions*, Profes-

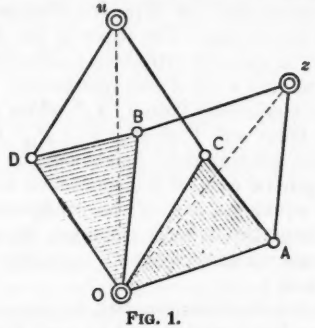

FIG. 1.

sor Emch has shown that all algebraic transformations of complex variables may be ealized by linkages. In this paper he describes a very simple linkage realizing the rotation $u = e^{i\theta} \cdot z$,

where θ is constant. In Fig. 1, all links with the exception of AC and BD are equal. If $AC = BD$ are chosen in such a manner that

$$\angle AOC = \angle BOD = \theta,$$

then it can easily be proved that $\angle zOu = \theta$, no matter how the linkage may be distorted. Thus, if z is moved into any part of the complex plane, it is clear that the point u represents the complex number $u = e^{i\theta} \cdot z$.

Taking $\theta = 2\pi/n$, $n-2$ equal cells may be successively combined in exactly the same manner, as $OCuD$ is attached to $OAzB$, thus forming a closed linkage of n cells. The vertices $z, u, u_1, \cdots$ of this linkage form a regular variable polygon of n sides. If

$$z = \sqrt[n]{w} = r^{\frac{1}{n}} e^{\frac{i\phi + 2k\pi}{n}} \qquad (k = 0, 1, 2, \cdots, n-1\,;\ w = re^{i\phi}),$$

then the other vertices of the polygon represent the remaining roots. This compound linkage has been devised to show in a simple case the realization of many valued functions by linkages, which in general presents considerable difficulties.

21. The plane connex of Clebsch is defined by equating to zero a double ternary form containing a set of point coördinates and a set of line coördinates. The analogous space connex, defined by forms involving the coördinates of a point and of a plane, has been studied by Krause (*Mathematische Annalen*, 1879), Sintsof and others. The paper of Dr. Kasner gives another extension to space by introducing forms involving a set of point coördinates and a set of line coördinates. The vanishing of such a quaternary-senary form $a_x^n A_P^m$, where x or $x_1 : x_2 : x_3 : x_4$ represents the point and P or $P_{12} : P_{13} : P_{14} : P_{34} : P_{42} : P_{23}$ the line, defines the general *point-line connex* of space. After a survey of the general case, of the connection with certain types of differential equations, and of the configurations defined by the intersection of two or more connexes, the author considers in detail the case where each set of variables is involved linearly, $n = 1$, $m = 1$.

To each point x there corresponds, by means of this bilinear connex, a linear line complex, and to each line P there corresponds a plane. The study of the connex consists in the discussion of these two transformations, which are both linear.

To enumerate a few of the results: The locus of the point whose corresponding linear complex is special is a quadric surface. There are five points, situated on this quadric, each possessing the property that the corresponding complex is special and has for its directrix a line passing through the point. There are two fundamental lines P, *i. e.*, such that the corresponding plane is indeterminate. The connex is defined geometrically by means of the (1, 1) correspondence which may be set up between the lines of a linear congruence and the points of a quadric surface. The paper concludes with the discussion of certain invariants and covariants and the establishment of normal forms with respect to both cogredient and digredient transformations.

22. Mr. Hook's paper, which will be published in the *Annals of Mathematics*, deals with curves given by either two or three equations of the form

$$x_i = \cos m_i(t + t_i) \quad (i = 1, 2 \text{ or } 1, 2, 3),$$

m_i and t_i being constants, x_i the cartesian coördinate, and t a variable parameter. The plane curves are classified into open and closed curves. The three-dimensional curves fall into four types based on the number of open or closed curve projections they possess on the three coördinate planes. This classification applies only to the case in which the ratios of the quantities m_i are commensurable. Transcendental curves are not considered at all. For each type of plane curves a formula is found for the number of real double points. On the space curves both double and quadruple points are possible and formulæ giving the numbers of these on each type of curve are found. A number of interesting theorems in regard to the relative positions of the multiple points on the curves are established. It is found that in all cases the tangents at the multiple points are real and distinct.

23. Dr. Eisenhart has applied the methods of Weingarten to the study of the infinitesimal deformation of the skew helicoid and has found that the solution is complete, for the characteristic partial differential equation of the second order can be integrated. The cartesian coördinates of the characteristic surfaces, which correspond to the helicoid with orthogonality of linear elements, are given by quadratures. They constitute a

general class of surfaces of which the surfaces of revolution form a subclass, corresponding to a constant value of one of the two arbitrary functions which enter into the general integral of the characteristic equation. When these surfaces are surfaces of revolution, and only in this case, their lines of curvature correspond to the asymptotic lines on the helicoid.

The associate surfaces of the deformation are given without quadrature and are found to be the general moulded surfaces of Monge. Moreover, when any moulded surface is given, the characteristic function which determines the corresponding deformation can be found at once. Surfaces of revolution are moulded surfaces and it is shown that when the associate surface is a surface of revolution the characteristic surface also is a surface of revolution, and conversely.

24. Professor Eiesland's paper contains a study of the null system

$$dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 = 0,$$

all the lines of which may by means of Lie's transformation

$$x_1 = \tfrac{1}{2}P_1,\ x_2 = X,\ x_3 = \tfrac{1}{2}P_2,\ x_4 = X_3,\ x_5 + x_1x_2 + x_3x_4 = X_5$$

be transformed into ∞^5 parabolas lying in planes parallel to the X_3-axis. The relations between the ∞^7 lines of the null system and these ∞^5 parabolas have been discussed. As an interesting application of this theory, a class of surfaces in ordinary space was found whose asymptotic lines are known. In the case of certain two-dimensional translation surfaces in the space M_5, a class of surfaces in ordinary space was obtained which Darboux arrived at in his Leçons, volume I, page 141, by an entirely different method.

In the last part of the paper the question of invariance of the null system in n-dimensional space is discussed; of all the euclidean motions $\left(\frac{n-1}{2}\right)^2$ rotations and a translation along an axis transform all the lines of the null system into themselves. For $n = 5$ the following theorems have been proved: 1° There exist ∞^{21} projective transformations which leave the null system invariant. 2° There exist ∞^{21} contact transformations leaving the differential equations

$$\frac{d^2X_3}{dx_1^2} = 0, \qquad \frac{d^2X_2}{dx_1^2} = 0$$

invariant. 3° There exist ∞^{10} projective transformations which leave invariant the complex of lines defined by the differential equations

$$dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 = 0,$$
$$dx_1dx_2 + dx_3dx_4 = 0.$$

This complex has been called an asymptotic complex on account of the rôle it plays in the transformation of surfaces in M_5 into surfaces of ordinary space on which the asymptotic lines are known; in fact, the lines of this complex have been shown, when transformed by Lie's transformation to become linear tangents along the asymptotic curves of the surface. 4° There exist only one rotation and a translation along the X_5-axis leaving an asymptotic complex invariant.

25. In Professor Roe's paper an elementary proof is given that any function $\phi(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ which satisfies the partial differential equation

$$\left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + \cdots + y_n\frac{\partial}{\partial x_n}\right)\phi = 0,$$

where the y's are independent of the x's, is a function of any set of $n-1$ independent determinants of the x's and y's of the form

$$(xy) = x_iy_{i+1} - x_{i+1}y_i.$$

The paper will be published in the *Annals of Mathematics*.

27. Mr. Ford's paper is concerned with the determination of sufficient conditions for a function $f(x)$ of the real variable x in order that each of the two following developments, occurring in mathematical physics, for $f(x)$ in terms of Bessel's function $J_\nu(x)$ may be differentiated term by term:

I. $$f(x) = 2\sum_1^\infty \frac{J_\nu(\lambda_n x)}{J_{\nu+1}^2(\lambda_n)}\int_0^1 xf(x)J_\nu(\lambda_n x)dx,$$

where λ_n is one of the positive roots of the equation $J_\nu(x) = 0$;

II. $$f(x) = (2r+2)\int_0^1 f(x)x^{r+1}dx + 2\sum_1^\infty \frac{J_r(\lambda_n' x)}{J_r^2(\lambda_n')}\int_0^1 xf(x)J_r(\lambda_n' x)dx,$$

where λ_n' is one of the positive roots of the equation $xJ_v'(x) - vJ(x) = 0$.

The results obtained are summarized in the following:

Theorem: Each of the series I and II converges when $a' < x < b' (0 < a' < b' < 1)$ to the limit $f(x)$, and when differentiated term by term the resulting series converge for the same values of x to the limit $f'(x)$, provided that $v > -\frac{1}{2}$ and that the function $\phi(x) = x^{-v} f(x)$ satisfies the following conditions:

Condition A: $\phi(x)$ is continuous, or is made up of a finite number of a continuous portions throughout the interval $0 \leqq x \leqq 1$.

Condition B: $\phi(x)$ possesses a derivative $\phi'(x)$ which is continuous throughout the interval $a' \leqq x \leqq b'$, and in the intervals $0 \leqq x < a'$, $b' < x \leqq 1$ is either continuous or is made up of a finite number of continuous portions. Also, the function $\frac{\phi'(x)}{x}$ is finite in the neighborhood of the point $x = 0$.

Condition C: $\phi(x)$ possesses a second derivative $\phi''(x)$ which is finite throughout the interval $a' - \epsilon \leqq x \leqq b' + \epsilon$ (ϵ arbitrarily small and positive).

Condition D: $\phi(1) = 0$.

Moreover, when $-1 < v \leqq -\frac{1}{2}$ the above theorem is true if we require also that the function $|x^v f(x)|$ be adapted to integration in the neighborhood at the right of the point $x = 0$.

28. Defining two homogeneous binary forms $f(x, y)$ and $\phi(x, y)$, each of the nth order, as apolar when they are so related that the invariant $\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}\right)^n$ vanishes identically, Professor Holgate finds a linear construction of the third element in one of two apolar cubic forms when its two remaining elements and the three elements of the other form are given. If A_1, A_2, A_3, are the three root points of $f^3(x, y) = 0$ and B_1, B_2, B_3, the three root points of $\phi^3(x, y) = 0$, then if B''' is the harmonic conjugate of B_1 with respect to A_1 and A_2, and C''' the conjugate of A_3 in the involution determined by the pairs A_1, A_2; B_1, B'''; similarly if B' and B'' are the harmonic conjugates of B_1 relative to the pairs A_2, A_3 and A_3, A_1, respectively, and C' and C'' the conjugates of A_1 and A_2 in the involutions A_2, A_3; B_1, B', and A_3, A_1; B_1 B'', respectively; then B', C'; B'', C''; B''', C''' is an involution such that any pair together with B_1 constitutes a triad apolar to A_1, A_2, A_3. Hence B_2 and B_3 need only be a pair in this involution, and if either of

these points is given the other can be readily found by linear constructions.

29. In the theory of continuous groups of transformations the collineation groups are of prime importance. In Lie's works are to be found complete lists of continuous groups of collineations in the plane, but nowhere does he give a list of all such groups in space of three dimensions. Professor Newson's paper contains a list of 490 continuous groups of collineations in space; this list is thought to be complete. A synthetic method was employed in determining these groups which is wholly independent of Lie's methods. These 490 groups are classified according to the thirteen well-known types of collineations in space and each is designated by a convenient symbol. This table is a natural extension of the table of groups of collineations in the plane given in Professor Newson's memoir in the *American Journal of Mathematics*, Volume 24, Number 2.

30. The group of collineations leaving a quadric surface F invariant is one of the most important subgroups of the general projective group. Miss Brewster's paper contains an exhaustive study of the ∞^6 collineations in the group $G_6(F)$. These are included under the types I, III, VI, IX, X and XI (Professor Newson's classification) and one special collineation of type XII. These collineations combine to form twenty-five continuous groups and seven mixed groups. These groups are classified according to their types, their invariant figures determined and the structure of each group discussed with reference to continuous groups of other types and singular transformations. A table is given at the end showing each group, its symbol, its invariant figure and the corresponding group in Lie's table on pages 251–254 of the third volume of "Theorie der Transformationsgruppen."

31. Dr. Pell applied a method given by Darboux (Leçons, etc., volume I, page 53 ff.) to determine the surfaces whose geodesics can be represented upon the plane as two-parameter curves of the form

$$f_1(x, y) + \mu f_2(x, y) + \nu f_3(x, y) = 0.$$

The problem has been treated by Beltrami's method by Dr. Stecker in the *Transactions* of 1901–1902.

32. The paper of Professor Heffter, which will appear in the *Transactions*, extends to space of m dimensions considerations concerning curvilinear integrals previously developed by him in the *Göttinger Nachrichten* of February 8, 1902, for the case $m = 2$ of the plane. As the curve C of integration is taken the general rectifiable curve; in case the curve C lies within a closed continuous region of continuity of the integrand function, the usual definition of the integral receives a certain modification, in that the range of values of the limitand sum is enlarged; this modification is seen to simplify the development of the theory as regards the so-called fundamental theorem of the integral calculus, the theory of the curvilinear integral as function of the upper limit in case it is independent of the path C, and the theorem of Cauchy.

EDWARD KASNER,
Assistant Secretary.

COLUMBIA UNIVERSITY.

THE MEETING OF SECTION A OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, PITTSBURGH, PA., JUNE 28–JULY 3, 1902.

BOTH in attendance and in the number and character of the papers presented, the Pittsburgh meeting of Section A of the American Association for the Advancement of Science, was very successful. The Secretary has not preserved a memorandum of the members who were present, but he estimates that forty or more attended the different sessions. The Section met on Monday afternoon, June 30, the Vice-President, Professor G. W. Hough, of Northwestern University, presiding, to hear the address of the retiring Vice-President, Professor James McMahon, of Cornell University. In the absence of Professor McMahon, his address, the subject of which was "Some recent applications of function theory to physical problems," was read by Professor R. S. Woodward, of Columbia University. This address was published in *Science* for July 25.

Amongst the papers on the programme of the Section were three important reports: I, "A report on quaternions," by Professor Alexander Macfarlane, which will be published in the *Proceedings* of the Association. II, "A second report on

recent progress in the theory of groups of finite order," by Professor G. A. Miller, Stanford University, which was presented in abstract by Dr. W. B. Fite, Cornell, in Professor Miller's absence. This report appears in the present number of the BULLETIN. III, "A report on the theory of collineations," by Professor H. B. Newson, the University of Kansas, which was read by title, in Professor Newson's absence. This report will be published in the *Proceedings* of the Association. Titles and abstracts of the other papers follow below.

Professor David P. Todd, Amherst College Observatory, presented two papers:

I. "On the adaptability of the glycerine clock to the diurnal motion of astronomical instruments, particularly those used in photographing solar eclipses."

The glycerine clock is an accurately constructed cylinder, about four inches in diameter, in which travels a piston, the rate of flow of the glycerine being controlled by a series of needle valves. By attaching a mirror or objective to a frame, equatorially mounted, and setting the clock under one of its arms at any convenient distance from the axis, the requisite rate for counteracting the diurnal motion of the sun can be given by means of the needle valves. This permits very heavy weights to be thrown on the piston and thereby the vibration of the instrument by wind can be precluded. When the run of the piston is finished, the glycerine is pumped out of the top of the cylinder and forced back into the bottom, and the run is commenced over again.

II. "On a convenient type of finder for very large equatorials."

In equatorials above twenty inches in aperture, the ordinary finder is necessarily mounted so far away from the axis of the great telescope that its use occasions much inconvenience, on account of the distance of its eye piece from that of the great tube. To obviate this difficulty Professor Todd proposes to construct the finder with a pair of reflectors, either planes or prisms, set at 45°, and to mount its tube in rings or bearings. By turning the tube in these, the finder's eye piece can be brought as near the eye piece of the great tube as is desired, or pushed away from it to admit attachment or adjustment of subsidiary apparatus.

"Series whose product is absolutely convergent," Professor Florian Cajori, Colorado College.

This paper is a continuation of the subject as developed by the author in his previous papers (*Transactions of the American Mathematical Society*, volume 2, pages 25–36 (1901); *Science*, volume 14, page 395 (1901), BULLETIN, volume 8, pages 231–236 (1902), and in the article of Alfred Pringsheim (*Transactions of the American Mathematical Society*, volume 2, pages 404–412 (1901). Some of the results previously obtained, relating to absolutely convergent products of two or more series, are generalized and the method of treatment is simplified. The construction of pairs of divergent series with real or complex terms is given, such that the product of the two series is not only absolutely convergent, but equal to any desired value, including zero.

Professor George Bruce Halsted, University of Texas, presented two papers:

I. "A new treatment of volume."

By a sect calculus, the product of two sects is fixed. The area of a triangle is defined as half the product of base and altitude, which is proved independently of the side chosen as base. Then the volume of a tetrahedron is defined as one third the product of base and altitude, which is proved independently of the side chosen as base. Then is proven the theorem: The volume of any tetrahedron is equal to the sum of the volumes of all the tetrahedra which arise from the first by making successively a set of transversal partitions. Then the theorem: However a tetrahedron is cut by a plane, this partition can be obtained in a set of transversal partitions using not more than two other planes. Then is proven as a theorem: If a tetrahedron T is in any way cut by any planes into a certain finite number of tetrahedra T_k, then is always the volume of the tetrahedron T equal to the sum of the volumes of all the tetrahedra T_k. Then the volume of any polyhedron is defined as the sum of the volumes of any set of tetrahedra into which it is cut, and this is proved independently of the set taken. A corollary is that if a polyhedron be cut into polyhedra, the sum of their volumes equals its volume. Then a prismatoid is defined, and by cutting it into tetrahedra its volume is proved to be

$$V = \frac{a}{4}(B + 3S) \text{ [see Halsted's Elements]}.$$

All the forms of elementary geometry follow as special cases

of this. For example, the volume of the cube with edge 1 is thus *proved* to be 1. Thus volume is completely treated without ratio, without continuity, without limits, without infinity or any infinite process.

II. "A new founding of spherics."

This founds two-dimensional spherics and deduces all its theorems by assuming three sets of postulates. It makes no use of euclidean space, no use of straight line, center, radius, no use of any theorem of solid geometry. Thus it is an establishing of two-dimensional Riemannian geometry. The second group of assumptions, the "betweenness" assumptions, are entirely new. There is no assumption that the shortest path between two points is on a straightest line or great circle. It is rigorously proved that two sides of a spherical triangle are together greater than the third. Thus is eliminated the objectionable assumption analogous to "the straight line is the shortest distance between two points."

"A new solar attachment," Professor Herbert A. Howe, Chamberlin Observatory, University Park, Colorado.

The purpose of this paper is to call the attention of teachers of astronomy to a new solar attachment, the "Shattuck solar," which is capable of illustrating the uses of a number of astronomical instruments. When it is attached to an ordinary engineer's transit, a teacher may give his students a good deal of practice with it in some of the fundamental processes of practical astronomy, without having recourse to expensive astronomical instruments. The instruments illustrated are the prism transit in the meridian, the zenith telescope, the prime vertical transit, the sextant, and the equatorial coudé.

"On the periodic solutions of the problem of three bodies," Professor E. O. Lovett, Princeton University.

Lagrange found five exact solutions of the problem of three bodies in each of which the bodies preserve an unvarying configuration which revolves with a uniform velocity. When the third body is of infinitesimal mass compared with the other two, it can describe small periodic orbits in the vicinity of the points where exact solutions exist. The latter points were called centers of libration by Gyldén, and Darwin calls the infinitely small body an oscillating satellite. Hill pointed out the fertility of the notion and made a splendid application of it

in his lunar theory. Poincaré elaborated the mathematical theory in his celebrated researches, and we owe to Darwin an extended collection of examples of periodic orbits.

One of the most recent investigations of such orbits is a suggestive paper by Charlier, in No. 18 of the *Meddelanden fran Lunds Astronomiska Observatorium*. In the *Monthly Notices of the Royal Astronomical Society* for November, 1901, Plummer has discussed some of Charlier's results in a more general manner.

It is the object of Professor Lovett's paper to determine the imaginary centers of libration and their corresponding orbits, and thus complete the analytical solution proposed by Charlier. The results cannot be expected to fit the sky, but they may be of some interest to mathematical astronomers. It appears that there are real periodic orbits corresponding to imaginary centers of libration.

"The rate of the Riefler sidereal clock, No. 56," Professor Charles S. Howe, Case School of Applied Sciences.

In order to obtain the best rate from a clock, it is necessary to have it under constant pressure and constant temperature. Constant temperature may be obtained in a clock room. Dr. Riefler, of Munich, is the only manufacturer in the world who makes a clock in an air-tight glass case. There are three of these clocks in the United States, one of which is at the Case Observatory. The mean daily rate of this clock for a trifle over three months was .116 of a second. The average daily variation from this mean daily rate was .015 of a second, and the maximum variation was .022 of a second. It is believed that these rates are the best ever published.

"A representation of the coördinates of the moon in power series which are proved to converge for a finite interval of time," Dr. F. R. Moulton, University of Chicago.

In the construction of a lunar theory three conditions are very desirable, if not absolutely essential, viz.: (*a*) that every step and the general process shall be proved to be logically valid, at least under certain conditions which are fulfilled by the moon; (*b*) that the labor of constructing the series shall not be so great as to be impracticable, and (*c*) that the arbitrary constants shall enter in such a way that they may be determined from observations with a high degree of precision.

The first condition has not been insisted upon heretofore. The nearest approach is in Delaunay's theory in which every one of the finite number of steps taken is separately valid, but in which the infinite sequence which is involved has not been shown to converge. Dr. Moulton imposes everywhere condition (*a*), and also (*b*) and (*c*) so far as possible.

By making use of Cauchy's existence theorem for the solutions of total differential equations and Mittag-Leffler's recent splendid researches on the representation of analytic function in extended areas by generalizations of power series, and by defining conveniently a certain anchor ring, Dr. Moulton arrives at a method of obtaining expressions for the coördinates of the moon which are valid for any length of time chosen in advance; and, moreover, it is possible to determine in advance a number of terms which will give an arbitrary degree of accuracy. These very general results fulfill condition (*a*), but unfortunately do not fulfill (*b*) and (*c*).

Another method is given depending upon the expansion of the expressions for the coördinates as power series in certain appropriately chosen parameters, the coefficients in the case of the linear variables being trigonometric functions of the time, and in the case of the angular variables, linear and trigonometric functions of the time. It is proved that these series converge for all values of the time not too remote from the initial epoch. When a sufficient number of terms have been taken so that the error committed shall be less than an arbitrary amount they may be rearranged as a Fourier series so that the expressions will be of the form given in the standard methods. This method fulfills all three of the conditions, but the second not so well as that first developed by Hill in his remarkable researches.

"The mass of the rings of Saturn," Professor Asaph Hall, South Norfolk, Conn.

The mass of these rings was first determined by Bessel in 1831 from the motion of the apsides of the orbit of Titan. This motion is about half a degree in a year. But the action of the figure of the planet, and the attractions of the other satellites were neglected; and, as Bessel pointed out, the resulting mass of the rings was too great. This mass is 1/118, the mass of Saturn being taken as the unit.

In this paper an equation was formed containing two indeter-

minate quantities depending on the figure of the planet, the mass of the rings, and the masses of the three brighter satellites, Rhea, Dione and Tethys. The small resulting action of the other satellites was estimated. The coefficients of these six indeterminate quantities can be computed with sufficient accuracy. The uncertainty in finding the mass of the rings arises chiefly from the lack of good values of the masses of the satellites. These masses must be found from the mutual perturbations of the satellites. Substituting the values of the masses of the satellites determined by Professor H. Struve, the principal coefficient depending on the figure of the planet was assumed to be 0.0222. The mass of the rings is 1/7092. It is probable that Struve's masses of the satellites are too small, and the above mass of the rings would be too great.

Saturn will soon return to our northern skies, and it is hoped that further observations and their discussion will give good values of the constants of this interesting system.

Professor John A. Eiesland, Thiel College, presented two papers:

I. "On a class of real functions to which Taylor's theorem does not apply."

As an example of a function to which Taylor's expansion does not apply Cauchy gave $f(x) = e^{-1/x^2}$. Pringsheim has the following series

$$f(x) = \sum \frac{(-1)^\nu}{\nu!} \frac{1}{a^\nu x + 1}, \qquad a > 1 \text{ and real},$$

which expanded gives rise to the series

$$\phi(x) = e^{-1} - e^{-a}x + e^{-a^2}x^2 - \cdots \pm e^{-a^m}x^m \mp \cdots$$

This series is convergent throughout the finite portion of the plane, while $f(x)$ in the neighborhood of $x = 0$ has an infinite number of poles $x = 0$. There is a class of functions of this kind which are distinguished by the additional property that they give rise to identically the same expansion. Such a function is

$$f(x) = \frac{\sum (-1)^\nu \left(\frac{\pi}{2}\right)^{2\nu+1}}{(2\nu+1)!} \cdot \frac{1}{1 + a^{2\nu+1}x},$$

which expanded becomes

$$\phi(x) = \sin\frac{\pi}{2} - \sin\frac{a\pi}{2}x + \sin\frac{a^2\pi}{2}x^2 - \cdots \qquad (a > 1).$$

If a is any even integer,

$$\phi_1(x) = 1,$$

if a is an odd integer of the form $4n + 1$,

$$\phi_2(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{x+1},$$

and if a is an odd integer of the form $4n - 1$,

$$\phi_3(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Hence all functions $f(x)$ having *different* poles may give rise to the same expansion in Taylor's series. The reason for this singular phenomenon must be sought in the periodicity of the coefficients in the expanded series. Another class of functions all having the *same poles* but differing in the values assigned to the coefficients present the same feature. Functions of the type

$$f(x) = \sum \frac{c_\nu}{1 + a_\nu x} \qquad (\lim a_\nu = \infty)$$

may be expanded in a power series $A_0 + A_1x + A_2x^2 + \cdots$ in which

$$A_0 = \sum_0^\infty c_\nu, \quad A_1 = \sum_0^\infty c_\nu a_\nu, \cdots, A_n = \sum_0^\infty c_\nu a_\nu^n.$$

Now this system of equations may be solved for the unknown c_ν by a method employed by Poincaré;* the system of values obtained is not unique, in fact as Poincaré shows

$$c_0 a^0, c_1 a, c_2 a^2, \cdots, c_\nu a^\nu, \cdots, -1$$

also form a system of solutions, and more generally

$$c_0 a^{0P}, c_1 a^{1P}, c_2 a^{2P}, \cdots, c_\nu a^{\nu P}, \cdots, (-1)^P$$

are solutions of the system. But to all the different functions $f(x)$ that may thus be constructed there corresponds only one

* *Bull. de la Soc. Math. de France*, vol. 8.

expansion $\phi(x)$ which of course does not represent any of the functions.

II. "On a class of transcendental functions with line singularities."

Borel has given the function

$$\phi(z) = \Sigma b_\nu x^{c_\nu} \qquad \left(\frac{c_\nu - c_{\nu-1}}{c_\nu} > N\right)$$

as an example of a class of functions which with all their derivatives remain finite and continuous within the unit circle as well as on it. To this class of transcendentals the following type may be added:

$$F(z) = \prod \frac{z-(1+a_\nu)e^{2\pi i\nu a}}{z-(1+b_\nu)e^{2\pi i\nu a}},$$

where $\lim a_\nu = 0$, $\lim b_\nu = 0$, and $a =$ an incommensurable number. The circle of radius unity is an essential singular line for all such functions, and this with respect to zeros as well as poles which are situated outside the circle, this curve being the point limit of poles and zeros. A more general type is

$$\Phi(z) = \prod \frac{z-(1+a_\nu)e^{\nu i a}}{z-(1+b_\nu)e^{\nu i a}} e^{\psi_\nu(z)},$$

where

$$\psi_\nu(z) = \frac{(a_\nu - b_\nu)e^{\nu i a}}{z-(1+b_\nu)e^{\nu i a}} + \tfrac{1}{2}\left[\frac{(a_\nu - b_\nu)e^{\nu i a}}{z-(1+b_\nu)e^{\nu i a}}\right]^2 + \cdots + \frac{1}{\nu-1}\left[\frac{(a_\nu - b_\nu)e^{\nu i a}}{z-(1+b_\nu)e^{\nu i a}}\right]^{\nu-1}.$$

All these functions can be proved finite and continuous within as well as on the unit circle together with all their derivatives up to any order as high as we please. They admit of no analytical expansion beyond this circle; the zeros and poles approach the unit circle in a narrowing spiral, but none of these singularities are situated on the curve.

"On a general method of subdividing the surface of a sphere into congruent parts:" Mr. Harold C. Goddard, Amherst College.

The problem was incidental to the practical problem of constructing a steel sphere one hundred feet in diameter, in con-

nection with a new method of mounting a telescope, as outlined in an article in the *American Journal of Science* for June, 1902, by Professor David P. Todd.

If a regular dodecaedron be inscribed in a sphere, planes determined by the center and each edge of the dodecaedron cut out on the sphere twelve equal regular spherical pentagons. If the vertices of each pentagon be connected with its center by arcs of great circles the surface of the sphere is divided into sixty congruent isosceles spherical triangles, whose angles are determined as 60°, 60° and 72°.

Professor J. Burkitt Webb, Stevens Institute of Technology, presented two papers:

I. "A possible new law in the theory of elasticity."

II. "Displacement polygons."

The first of these papers was presented by abstract, and the second by title. The law referred to in the first is: "If the forcible change of the distance between two points in an elastic system changes the distance of two other points by a certain amount, then the same force applied to alter the distance of the two other points will change the distance of the first two points by the same amount."

"On extracting roots of numbers by subtraction," Dr. Artemas Martin, Washington, D. C.

This paper is a development and extension of an article, published in the *Philosophical Magazine* for September, 1880, by the Rev. F. H. Hummell. Dr. Martin shows how to determine any root of any perfect power by subtracting from the given power a certain series of subtrahends. If the given number is not a perfect power that fact is revealed at a close of the operation. The last remainder is always equal to the root sought, if the given number is a perfect power, and the number of subtractions is less by unity than the root sought. The general formula for the nth subtrahend of the mth root is

$$S_n = (n+1)^m - (n^m + 1),$$

or
$$S_n = S_{n-1} + (n+1)^m + (n-1)^m - 2n^m.$$

The paper will be published in the *Mathematical Magazine*.

"On the positions of the northern circumpolar stars:" Professor Milton Updegraff, Washington, D. C.

The questions discussed by Professor Updegraff were:

(1) The importance of accurate knowledge of the positions of the fixed stars, and the greater attention, comparatively speaking, which has hitherto been paid to the places of the equatorial stars.

(2) The fundamental work on the circumpolar stars of the Greenwich and Pulkowa observatories. Differential work done by Groombridge, Carrington and Schwerd. Later work done at Greenwich and Pulkowa, and in America. Observations of circumpolar stars requested by Auwers.

(3) By observing the circumpolar stars at both upper and lower culminations on the same day, their places can be determined so that the results are practically fundamental. The fainter stars can be observed at both culminations on the same day, during the fall and winter months in middle latitudes, by observing a certain list of stars in the early morning and again early in the evening. Instrumental errors are largely eliminated, as also are errors of personal equation and of the refraction tables. The periodic variation of the position of the pole may be determined and also eliminated from the observed places. The formula for reducing the observations is the following well known modification of Bessel's formula for the reduction of observations in right ascension,

$$a = T + \Delta T + m + (n + e)\tan\delta + c(\sec\delta - \tan\delta).$$

(4) The desirability was pointed out of an extended series of fundamental observations of both equatorial and circumpolar stars for increasing, for example, our knowledge of the motions and distribution of the stars in space. Such a series of observations would be useful in constructing a general catalogue of fundamental stars, numbering several thousand, and selected with reference to their suitability to serve as standard points of reference.

(5) In conclusion certain advantages in the location of the Naval Observatory in Washington for fundamental work on the fixed stars were stated. Washington is twelve and one half degrees in latitude south of Greenwich, twenty degrees south of Pulkowa, and seven and one half degrees south of the new branch of the Pulkowa Observatory at Odessa. It is however far enough north so that the northern circumpolar stars can be advantageously and accurately observed. For these reasons it may be said that perhaps no better location, as

regards latitude, could be found for supplementing the fundamental work done at Greenwich, Pulkowa and the Cape of Good Hope. The climate at Washington is very good for work of this kind. The local conditions are also favorable and are likely to remain so.

"The definite determination of the causes of variation in level and azimuth of large meridian instruments," Professor G. W. Hough, Dearborn Observatory, Evanston, Ill.

Professor Hough went into an elaborate discussion of several series of observations upon the causes and effects of this variation with different styles of mounting. His conclusion was that stone piers give the best results. The paper gave rise to some spirited discussion.

"Some theorems on ordinary continued fractions," Professor Thomas E. McKinney, Marietta College.

Let D be any positive integer not a perfect square and let its square root be represented by an ordinary continued fraction. Professor McKinney determines the form of D so that the continued fraction representing its square root may have a period with one, two, three or four elements and applies the results to the determination of the number of reduced forms in the class to which the indefinite quadratic form $(1, 0, D)$ belongs.

"On the forms of sextic scrolls of genus one," Dr. Virgil Snyder, Cornell University.

Dr. Snyder applied the same methods to find the forms of sextic scrolls of genus 1 as he previously employed in the study of the unicursal surfaces. Thirty-three types are found, ten of which have a multiple conic. The same method is being applied to sextic scrolls of genus greater than 1. The complete paper will be published in the *American Journal of Mathematics.*

"Transformation of the hypergeometric series," Professor Edgar Frisby, U. S. Naval Observatory.

If in the differential equation of the second order connecting the elements of the hypergeometric series

$$P = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha\beta(\alpha+1)(\beta+1)}{1\cdot 2\cdot\gamma\cdot(\gamma+1)}x^2 + \text{etc.}$$

$x^m P'$ be substituted for P, new relations are obtained in which

P' takes the place of P, and the new elements are functions of the original elements. μ is determined from the condition that the new series must be of the same general form as the old. If, in addition, x be replaced by $1/x$ another series is obtained. From these two new series, by proper substitution of the new derived elements, are obtained, almost by inspection, the twenty-four different series ordinarily given in works on differential equations.

EDWIN S. CRAWLEY,
Secretary.

UNIVERSITY OF PENNSYLVANIA.

SECOND REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

BY PROFESSOR G. A. MILLER.

(Read before Section A of the American Association for the Advancement of Science, Pittsburg, July 2, 1902.)

THE main extensive treatments of this theory which have appeared during the four years since my first report was presented before this Section are: The articles in volume I of the Encyklopädie der mathematischen Wissenschaften on "Endliche discrete Gruppen," "Galois'sche Theorie mit Anwendung," and "Endliche Gruppen linearer Substitutionen," by Burkhardt, Hölder, and Wiman respectively; Weber, Lehrbuch der Algebra, second edition, volume 2, 1899; Bianchi, Lezioni sulla teoria dei gruppi di sostituzioni, 1899;* Echegaray, Lecciones sobre resolucion de ecuaciones y teoria de ecuaciones, 1899;† Netto, Vorlesungen über Algebra, volume 2, 1900; Pierpont, "Galois theory of algebraic equations,"‡ 1900; Dickson, Linear groups, 1901; Burnside and Panton, Theory of equations, volume 2, 1901.

In the present, as in my first report, it is my intention to avoid, as far as practicable, the consideration of those recent advances which have received considerable attention in these

* Printed edition of the work lithographed in 1897.
† Reviewed in *L'Enseignement Mathématique*, vol. 2 (1900), p. 227.
‡ *Annals of Math.*, 2d series, vols. 1 and 2.

articles and treatises, and to devote most of the space to a few recent developments which seem to offer inviting fields of investigation. It is hoped that this report will soon be supplemented in some directions, like the first report the usefulness of which was greatly extended by Professor Dickson.*

For several years Dr. Steinitz, of Charlottenberg, has had under consideration a report on the theory of groups of finite order for the Deutsche Mathematiker-Vereinigung. In a recent letter he expressed the hope of having this report ready for the next meeting of the Vereinigung, which will be held at Karlsbad during the coming September in connection with Section I of the Gesellschaft Deutscher Naturforscher und Aerzte. Dr. Easton, of the University of Pennsylvania, has prepared a bibliography of the theory of substitutions together with a collection of theorems and definitions, which is now in press.

§1. Abstract Groups.

If the three numbers l, m, n are the orders of three operators one of which is a product of the other two, it is well known that the group generated by any two of these operators is completely determined by l, m, n provided at least two of these numbers are 2; or one is 2, another 3, while the third is 3, 4, or 5. The groups which may be defined in this way, together with those which are cyclic, are known as the groups of genus zero, and have received a great deal of attention as they are the groups of the rotations into themselves of the regular solids and hence enter prominently into many geometric considerations.

It has recently been proved † that three operators L, M, N whose orders are fixed numbers l, m, n respectively can always be so selected as to generate any one of an infinite system of groups of finite order whenever each of these fixed numbers exceeds unity and the three do not satisfy one of the special conditions mentioned in the preceding paragraph. The proof of this theorem is based upon the fact that it is always possible to find three operators such that one of them is a product of the other two and that their orders are the three arbitrary numbers l, m, n. Both of these theorems were proved by means of sub-

* Dickson, "Report on the recent progress in the theory of linear groups," Bulletin, vol. 6 (1899), p. 13.

† *Amer. Jour. of Math.*, vol. 24 (1902), p. 96; cf. Dyck, *Math. Annalen*, vol. 20 (1882), p. 34.

stitutions. It would appear desirable that more abstract proofs should be given, especially since the published proofs are based upon considerations which are not commonly employed.

The infinite systems of groups which belong to a particular set of values of l, m, n have received very little attention except in the three cases where these numbers are such as to give rise to groups of genus one. These cases have been studied geometrically by Dyck * and Burnside.† Recently they have been studied from the purely analytic standpoint and some interesting common properties have been found.‡ Perhaps the most important of these is the fact that all of them contain an abelian commutator subgroup.

The groups of genus one which are not included in the three cases mentioned in the preceding paragraph are those which may be generated by three operators of order two whose continued product is also of order two, but which cannot be generated by two operators of order two. All of these groups may be constructed by adding to *any* abelian group A, having just two independent generators, an operator of order two which transforms each operator of A into its inverse. In this way a group is obtained whose order is twice the order of A and which contains only operators of order two besides those of A. This is the simplest category of groups of genus one and has many properties in common with the dihedral rotation groups. It may be observed that none of these groups can be generated by less than three operators. This category includes only one abelian group, viz., the group of order 8 and of type (1, 1. 1).

From what precedes it is clear that little is known about the groups generated by the two operators L and M when the three numbers l, m, n are given. A closely related problem which has received but little attention is the study of the groups in which the order of every operator is a divisor of a fixed number k. The simplest case, viz., when $k = 2$, has been completely solved.§ There is just one such group of order 2^α for every value of α. This is the only value of k for which all the possible

* *Loc. cit.*

† Theory of groups of finite order, 1897, p. 293.

‡ *Quar. Jour. of Math.*, vol. 33 (1901), p. 76; *Annals of Math.*, vol. 3 (1901), p. 40; The case when the groups are generated by two operators of order 4 whose product is of order 2 was considered by Manning in a paper read at the first meeting of the San Francisco section of the Amer. Math. Soc., May, 1902.

§ *Quar. Jour. of Math.*, vol. 28 (1896), p. 208.

groups are abelian. When $k = 3$ it is very easy to prove that all the conjugate operators are commutative. Moreover, it is evident that this property is not common to all the groups for any value of k greater than 3.

Recently Burnside has made a study of all the possible groups when $k = 3$, and also of the special case when $k = 4$ and the groups have but two generators.* The more general case when k is any prime p seems to present considerable difficulty. The order of such a group is evidently a power of p. In the special case when there is an abelian subgroup whose order is the order of the entire group divided by p it is not difficult to determine all the possible groups.† Even for p^5, $p > 3$, there are groups which contain only operators of order p without containing the abelian group of order p^4.

The questions considered in the preceding two paragraphs may be regarded as special cases of the problem to determine all the groups which are conformal with systems of abelian groups. If two abelian groups are conformal, *i. e.*, if they contain the same number of operators of each order, they cannot be distinct; but with respect to the non-abelian groups the matter is entirely different. Recently all the possible abelian groups which are conformal with non-abelian groups have been determined,‡ but the total number of non-abelian groups which are conformal with fixed types of abelian groups are known in only a few cases. For instance, it is known that there is only one non-abelian group which is conformal with the abelian group of order p^m and of type $(m-1, 1)$. The only other types of order p^m for which all the conformal non-abelian groups are known are $(m-2, 2)$ and $(m-2, 1, 1)$.§

If p^α is the highest power of the prime number p which divides the order of a group then the group must contain $1 + kp$ conjugate subgroups of order p^α according to Sylow's theorem. Moreover, it is known that there are groups for all values of p when k is either 0 or 1. Burnside has recently examined the cases when k is either 2 or 4 and found that in both of these cases the possible values of p are greatly restricted. His results are, when $k = 2$, then $1 + 2p$ is either a prime or a power of 3; and when $k = 4$, then $1 + 4p$ is either a prime, 9,

* Burnside, *Quar. Jour. of Math.*, vol. 33 (1902), p. 230.
† BULLETIN, vol. 8 (1902), p. 39.
‡ BULLETIN, vol. 8 (1902), p. 154.
§ *Loc. cit.*, p. 206.

or a power of 5. The case when $k = 3$ seems to present greater difficulties but it is easy to see that there are also limitations on the form of p for this value of k.*

Most of these advances in the theory of abstract groups have been made by means of substitution groups. When this method is employed comparatively little use is made of definitions, as all the steps can be conveniently illustrated by means of concrete examples. For some purposes it is however desirable to proceed more abstractly. This seems to be especially the case in a systematic presentation of the subject. Hence it becomes desirable that the definitions should be so formulated as to be as useful as possible. It is also desirable to avoid redundancies as these mar the beauty of the presentation. Recently Dr. Huntington has pointed out redundancies in the definitions usually given and has formulated definitions which do not involve these objectionable features. † Professor Moore has also presented a note on this subject.‡

Scarpis has recently considered some properties of commutators and of commutator subgroups. He proves by a method which differs somewhat from the one employed earlier, that the important question of solvability can readily be decided by means of commutator subgroups.§ Although he gave a general reference to the earlier article in which this theorem is proved, yet one would naturally infer in reading his article that he considered the theorem as new. One might also question the propriety of naming these groups after Dedekind, since he published very little in regard to them, and, moreover, was not the first to make their properties known.‖

The interesting question whether every operator of a commutator subgroup is a commutator has been answered in the negative in volume 2, page 133, of the second edition of Weber's Algebra. Dr. Fite has found some conditions which are sufficient for the existence of such operators, and has published a group which satisfies these conditions.¶ Wundt proved a very interesting theorem in regard to the special class of soluble groups which possess a principal series of composition such that

* Burnside, *Messenger of Mathematics*, vol. 31 (1901), p. 77.

† Huntington, BULLETIN, vol. 8 (1902), p. 296.

‡ BULLETIN, vol. 8 (1902), p. 373.

§ Scarpis, *Giornale de Matematiche*, vol. 39 (1901), p. 376; cf. BULLETIN, vol. 6 (1899), p. 105.

‖ Cf. Frobenius, *Berliner Sitzungsberichte*, 1896, p. 1348; Cantor, *Geschichte der Mathematik*, vol. 2 (1900), p. 811.

¶ BULLETIN, vol. 6 (1900), p. 181.

all of the corresponding factor groups are of prime order. Whenever this condition is satisfied he proved that the commutator subgroup must be the direct product of groups whose orders are powers of primes.* This proof is partly based upon the fact that every group contains an invariant subgroup (which may be the identity) which is the direct product of groups whose orders are powers of primes and includes all other invariant subgroups that have this property.

The most extensive work on the enumeration of abstract groups which has appeared during the last few years seems to be the Enumeration des groupes d'opérations d'ordre donné, 1901, 128 p., lith., by Le Vavasseur. This work, which is published by Hermann, Paris, was noticed by the writer too lately to permit an examination before writing this report. The same author has recently published less extensive enumerations of the groups of order p^2q^2 and also those of order $16p$, p and q being primes and p being odd.† Western has considered the groups of order p^3q,‡ but no one seems to have determined all the groups in the remaining case where there are four prime factors, viz., when the order is p^2qr. The enumeration of groups of low orders has been extended to those of order 64.§

Dickson has embodied a large number of the results of his extensive investigations in his Linear Groups. The last chapter of this work is devoted to a summary of known systems of simple groups. It is observed that two of these systems involve simple groups of the same order that are not isomorphic. The object of some of his more recent papers is to show that several branches of group theory may be correlated by means of transformations in a given domain of rationality. The special simple group of order 504 has been defined more abstractly by Burnside in the *Mathematische Annalen*, volume 52 (1899), page 174. Fricke considers this group from a different standpoint in the same volume, page 321.

In a memoir published in *Bihang till Kongl. Svenska Vetenskaps Akademiens Handlingar*, volume 25 (1900), Wiman determined all the subgroups of a doubly infinite system of simple groups. The article begins with a study of the Galois imaginaries and the groups of finite order defined by congruences.

* Wundt, *Math. Annalen*, vol. 55 (1901), p. 479.

† La Vavasseur, *Comptes rendus*, Paris, vol. 128 (1899), p. 1152; vol. 129, p. 26.

‡ Western, *Proc. of the London Math. Soc.*, vol. 30, p. 209.

§ *Quar. Jour. of Math.*, vol. 30 (1898), p. 243.

Bauer, in a series of articles published in *Nouvelles Annales*,* proved a number of theorems relating to the number of subgroups of particular orders that can occur in a group whose order is known. Most of these theorems are based upon those proved by Frobenius in the *Berliner Sitzungsberichte* of 1895.

While some questions in regard to abelian groups, for instance those which relate to the number and the types of groups of a fixed order, have been completely solved yet there is a large number of others which have received but little attention. Steinitz has recently considered one of the latter, viz., the determination of how many times a group γ is contained in the product $\alpha\cdot\beta$ of two abelian groups. In other words, to determine the number of subgroups α' such that α' is isomorphic with α and γ/α' is isomorphic with β. Representing this number by $t(\gamma;\ \alpha,\ \beta)$ Steinitz investigated some of the properties of this function and determined all these groups when β is cyclic or has just two independent generators.†

On the occasion of the jubilee of Sir G. G. Stokes, Burnside presented a brief memoir on the groups of even order which contain no operator of even order except those of order 2. He found that these groups may be divided into three distinct classes which are represented by the dihedral, tetrahedral, and the icosahedral rotation groups. In a more recent paper communicated to the London Mathematical Society he considers the groups in which every two conjugate operators are permutable and proves that the necessary and sufficient condition that the order of such a group is finite is that the generators are of finite orders. The case when the order is a power of 3 is exceptional.‡

§ 2. Holomorphisms.

A simple isomorphism of a group G with itself has been called a holomorphism or an automorphism of G.§ The totality of the holomorphisms of G correspond to a group I known as the group of isomorphisms. When G is transformed by its own operators one or more holomorphisms are obtained. The totality of the holomorphisms that can be obtained in this way correspond to an invariant subgroup I_1 of I, which is known as

* Vol. 19 (1900), pp. 59, 508, 509.

† Steinitz, *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 9 (1901), p. 80.

‡ *Nature*, vol. 66 (1902). p. 71.

§ *Annals of Math.*, vol. 2 (1901), p. 78; Frobenius, *Berliner Sitzungsberichte*, 1901, p. 1324.

the group of cogredient isomorphisms. When I is abelian, G is said to be metabelian and vice versa. A characteristic property of metabelian groups is that all their commutators are invariant.

The metabelian groups constitute a special case of a category of groups which seems to have been first studied by Ahrens; viz., those in which one arrives at the identity by forming the successive groups of cogredient isomorphisms.* It is easy to prove that all the groups of this category are the direct products of groups whose orders are powers of single prime numbers and vice versa.† Dr. Fite has recently made an extensive study of the properties of metabelian groups and some of his results have been published in the form of abstracts. His memoir on these groups is to appear soon in the *Transactions of the American Mathematical Society*.

The term metabelian seems appropriate since these groups have many properties which differ but slightly from those of abelian groups. In the development of their theory it is convenient to make prominent use of the commutator subgroup and the group of cogredient isomorphisms. When the former is cyclic and of order p^α, p being any prime, the latter contains an even number of independent generators, and the corresponding metabelian groups are the direct products of abelian groups and a metabelian group of order p^m.‡ The Hamiltonian groups constitute a special case of those metabelian groups whose group of cogredient isomorphisms is the four-group. The theory of metabelian groups clearly involves the determination of all the abelian groups which can be groups of cogredient isomorphisms. Most of the results in this direction are found in the memoir mentioned in the preceding paragraph.

A number of important properties of a group are exhibited in its group of isomorphisms. Comparatively little progress has been made in the theory of the simple isomorphisms of a group with itself. One of the most important steps in this direction has been the determination of all the invariant operators of I when G is abelian. These operators correspond to the holomorphisms of G in which every subgroup corresponds to itself. If this condition is satisfied every operator of G corresponds to

* Ahrens, *Leipziger Berichte*, vol. 49 (1897), p. 616; cf. Fite, BULLETIN, vol. 8 (1902), p. 236.

† Burnside, Theory of groups, 1897, p. 115; cf. Loewy, *Math. Annalen*, vol. 55 (1901), p. 67.

‡ BULLETIN, vol. 7 (1901), p. 86.

the same power of itself.* Hence the number of invariant operators of I is the totient of the maximum order of any operator of G.

These results have been very materially extended by Mr. Young in his recent article "On the holomorphisms of a group." † He employs for the first time the convenient term *α-holomorphism* to denote a simple isomorphism of a group with itself such that every operator corresponds to its αth power, and inquires into the necessary and sufficient condition that a non-abelian group may admit α-holomorphisms besides the identical one. He finds that a group cannot admit an α-holomorphism unless the $(\alpha-1)$th power of every operator is invariant. When the order of the group is a power of a prime these α-holomorphisms correspond to invariant operators in the group of isomorphisms but it is not always possible to obtain all the invariant operators of the latter group in this manner.

In any holomorphism each operator corresponds to itself multiplied on the right by some operator of the group. The totality of these multiplying operators do not necessarily form a group. For instance, if the symmetric group of order 6 is transformed by one of its operators of order three, the holomorphism will be such that the multiplying operators do not form a group; but these operators form a group when the holomorphism may be obtained by transforming by an operator of order two. In every holomorphism of an abelian group A these multipliers always form a group ‡ and it is frequently convenient to consider the holomorphisms of A from this standpoint.

Not only do the multipliers of the preceding paragraph form a subgroup of A but this subgroup (which may coincide with A) is also isomorphic with A in every possible simple isomorphism of A with itself. If the order of this subgroup is either p or $2p$; the resulting holomorphism of A will correspond to an operator of order p, $2p$, or $\frac{p-1}{\alpha}$ (α being a divisor of $p-1$) in the group of isomorphisms I of A. If, moreover, each operator of this subgroup corresponds to itself in such a holomorphism, the order of the corresponding operator of I is clearly equal to the order of the largest operator of the subgroup. Very little has been done towards the development of the theory of holomor-

* *Transactions Amer. Math. Soc.*, vol. 2 (1901), p. 260.
† Young, *Ibid.*, vol 3 (1902), p. 186.
‡ BULLETIN, vol. 6 (1900), p. 337.

phisms from this standpoint. It may be possible to find simple conditions which are necessary and sufficient to establish holomorphisms of non-abelian groups in this manner. Such conditions would be of considerable interest.

One of the simplest ways of obtaining a holomorphism of a non-abelian group G is to make each of its operators A correspond to its transform with respect to any one of its operators B. In calling attention to a slight error in Burkhardt's article in the Encyklopaedie, Loewy writes this transform BAB^{-1} instead of $B^{-1}AB$.* Moreover Loewy writes $BA^{-1}B^{-1}$ instead of $BA^{-1}B$ as given by Burkhardt. The latter is, however, a very insignificant error; while the former cannot be regarded as an error since some good writers have used BAB^{-1} for the transform of A with respect to B;† but the majority of writers (including Jordan, Lie, Netto, Weber, Burnside, Bianchi) use the form $B^{-1}AB$. The form used by the minority seems to have decided advantages. For instance, in considering the expression

$$(AB)^n = AB\,AB\,AB\,AB\cdots$$
$$= A\,BAB^{-1}\,B^2AB^{-2}\,B^3AB^{-3}\cdots,$$

it is convenient to call BAB^{-1} the transform of A, especially if some of the commutators are not invariant. Although this is not a very important matter yet it would be of some interest to know which of these forms is the most advantageous, especially since transforms are so frequently employed.

If any group G is represented as a regular substitution group there is a totality of substitutions in the same elements as those involved in G which transform G into itself. These form a non-regular group known as the holomorph of G. Any one of the largest subgroups which do not involve one of the elements of this holomorph is simply isomorphic with the group of isomorphisms of G and every holomorphism of G can therefore be obtained by transforming G by some operator in such a subgroup. The substance of these remarks is found in my former report, page 245. It is repeated here since Burkhardt's statement on bottom of page 221 of the Encyklopaedie seems to imply that the holomorph of G is also its group of isomorphisms, which is evidently incorrect.

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* Loewy, *Math. Annalen*, vol. 55 (1901), p. 68.

† Dyck, *Math. Annalen*, vol. 22 (1883), p. 75. The present usage seems to be due to the influence of Jordan; cf. Hagen, Synopsis der höheren Mathematik, vol. 1 (1891), p. 285. Cauchy and Serret used the form BAB^{-1}, which was called the derivative of A by Betti.

It is known that cyclic groups are the only abelian groups whose groups of isomorphisms are abelian.* No one seems to have investigated the question whether a non-abelian group can have an abelian group of isomorphisms. If this question should be answered in the negative it will follow that the system of groups which Weber calls the most important example of abelian groups of finite order † is composed of all the possible commutative groups of isomorphisms. Since the group of cogredient isomorphisms of a non-abelian group cannot be cyclic, it follows that a cyclic group cannot be a group of isomorphisms unless its order is of the form $p^{\alpha}(p-1)$, p being an odd prime.

Groups which have the same group of isomorphisms may differ very much from each other. The totality of such groups has been determined in only a few cases while no one seems to have determined all those which have the same group of cogredient isomorphisms. Some of the simplest cases would evidently present very few difficulties. For instance, when the group of cogredient isomorphisms of G is of order p^2 it must be of type (1, 1) and G must be the direct product of abelian groups and a non-abelian group P of order p^m whose group of cogredient isomorphisms I_1 is of order p^2. Moreover, P contains an abelian subgroup of order p^{m-1} which includes the p^{m-2} invariant operators. When this abelian subgroup is cyclic it is well known that there is only one such group. When it is of type $(1, 1, 1, \cdots)$ there are clearly three such groups if $p > 2$ and $m > 3$; one is conformal with the abelian group of type $(1, 1, 1, \cdots)$ while the others are conformal with the one of type $(2, 1, 1, \cdots)$.

§ 3. Substitution Groups.

Maillet and Burnside have considered the transitive groups of degree n and of class $n-1$. In his thesis entitled, "Recherches sur les substitutions, et en particulier sur les groupes transitifs," Paris, 1892. Maillet divides these groups into two categories. The first of these included all those which contain a regular group of order n, while the second included those which do not have this property. He states that in all the particular cases which he has examined there is no primitive group belonging to the second category. In several later articles pub-

* *Transactions*, vol. 1 (1900), p. 397.
† Weber, Lehrbuch der Algebra, vol. 2 (1899), p. 60.

lished in the *Bulletin* of the French Mathematical Society * he extends a number of his results in regard to these groups.

Burnside takes up the same question both in his Theory of Groups and also in an article published in the *Proceedings of the London Mathematical Society*.† In the latter he proves that every transitive group of degree n and of class $n-1$ contains a regular group of order n whenever $n \geqq 81{,}000{,}000$. Recently Frobenius proved ‡ that this theorem is true for all values of n and thus established one of the most useful recent theorems in the theory of substitutions. Since the order of each operator in the regular group is a divisor of n while the order of every other operator divides $n-1$ it follows that there can be only one subgroup of order n.

During the last four years the enumerations of intransitive, imprimitive, and primitive groups have each been extended to one larger degree. The list of the intransitive groups of degree eleven includes 1,492 distinct substitution groups, which is about 500 more than the number for degree 10. § Nothing seems to have been done towards giving general formulas by means of which the number of these groups can be determined without trial except for the cases when the order is either the product of two distinct primes or when it is 4. Miss Martin published a list of the imprimitive groups of degree 15 and also one of the primitive groups of degree 18. ‖ She remarks that Dr. Kuhn has also made a study of the imprimitive groups of degree 15 and has arrived at results which differ greatly from those which she obtained.

The efforts to discover a simple group of odd composite order have led to the study of primitive groups of such an order, for every simple group is simple isomorphic with one or more primitive substitution groups. Burnside proved the interesting theorem that the maximal subgroup which does not involve one of the elements of such a group must contain an even number of transitive constituents.¶ Dr. Rietz has also established a number of theorems relating to this maximal subgroup** which seems to present one of the most accessible approaches towards

* Vol. 25 (1897), p. 16; vol. 26 (1898), p. 249.
† Vol. 32 (1901), p. 240.
‡ Frobenius, *Berliner Sitzungsberichte*, 1901, p. 1220.
§ Miller and Ling, *Quar. Jour. of Math.*, vol. 32 (1901), p. 342.
‖ Miss Martin, *Amer. Jour. of Math.*, vol. 23 (1901), p. 259.
¶ Burnside, *Proc. of the London Math. Soc.*, vol. 33 (1901), p. 165.
** BULLETIN, vol. 8, pp. 17 and 275.

the theory of primitive groups. By means of these theorems and the important theorem, due to Burnside, that there is no primitive group of odd composite order and of a prime degree p that contains more than one subgroup of order p, Dr. Rietz has determined all the primitive groups of odd order whose degree does not exceed 242. All these groups are solvable.

It is well known that the average number of elements in all the substitutions of a transitive group of degree n is $n-1$. Hence every transitive group must contain at least $n-1$ substitutions of degree n to make up the average for the identity. In general, for each of its substitutions of degree $n-\alpha$ there must be $\alpha-1$ substitutions of degree n. Dr. Rietz has recently proved * a closely related theorem to the effect that a primitive group of degree n and of order g contains more than $g/x+1$ substitutions of degree less than n, x being the number of transitive constituents in a maximal subgroup of degree $n-1$. This theorem is closely related to the investigations of Jordan, Bochert, Maillet, and others in reference to the class of a primitive group.

Although the primitive groups of low classes present much greater difficulties than those of low degrees, yet it is easy to determine all the primitive groups of a prime class p while no one has yet succeeded to determine those of any general degree.† In a memoir published in *Videnskabs-Selskabets Skrifter*, 1897, Sylow considers the groups of degree p and of order $p(p+1)\pi$, π being a divisor of $p-1$. He proved that there are no groups of degree p and of order $p/2(p^2-1)$ except in the cases when p has one of the values 5, 7, 11. In each of these special cases there is only one such group.

Lombardi has recently developed a number of theorems in regard to imprimitive groups.‡ Although his results differ but slightly from well-known theorems yet no references are given. Dr. Kuhn has made a much more extensive study of the theory of these groups. Only abstracts of his memoir have appeared.§ In the first of these it is pointed out that there are groups in addition to those noted by Dyck in *Mathematische Annalen*, volume 22, which cannot be represented in any non-regular imprimitive form. In connection with Wilkinson's article in regard to Jordan's theorem on the constancy of the factors of

* BULLETIN, vol. 8, p. 276.
† Jordan, *Liouville*, vol. 17 (1872), p. 363.
‡ Lombardi, *Giornale di Mathematiche*, vol. 39 (1901), p. 134.
§ BULLETIN, vol. 7 (1901), pp. 5 and 116.

imprimitivity * it should be observed that Jordan himself noted the error in his Traité des Substitutions. He published a note in regard to the matter in *Giornale di Matematiche*, volume 10 (1872), page 116.

An abstract group is generally said to be represented by a substitution group if the two groups are simply isomorphic. The term is used in this sense when it is said that every abstract group can be represented as a regular substitution group. Quite recently Burnside published an article in which he uses this term in a more general sense in accord with the phraseology used by Frobenius in the study of linear substitutions. Any substitution group to which an abstract group has an $(\alpha, 1)$ isomorphism is said to be a representation of the abstract group.†

According to this use of the term 1, (ab) is a representation of every group which involves a subgroup of half its order. It appears to the writer somewhat unfortunate to use a term which had such definite meaning in the theory of substitutions, in this general way. It may be added that some of the statements of this article are not in accord with known results unless other terms are also given an unusual meaning. Lines 8–11 on page 161 may serve as an instance. Additional references to other articles bearing on the same question would have increased the usefulness of the article very materially.

Beginners in the study of the theory of substitutions are naturally much hampered by the different uses of the same term as well as by the number of different terms employed to express the same concept. Within the last few years Netto has perhaps made the most serious contribution towards increasing the latter difficulties by using such new terms as conjug, conjuge complexe, and autojugen Theiler.‡ The last of these terms is used for invariant subgroup so that the Germans can now express this concept by any one of the following six terms: invariante, ausgezeichnete, or monotypische Untergruppe; autojuger, eigentlicher, or Normaltheiler. The first four and the last of these seem to be due to König, Lie, Frobenius, Netto, and Weber respectively, while eigentlicher Theiler seems to have been derived from décomposition propre, used by Galois § who introduced the concept of invariant subgroup.

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* Wilkinson, *Quar. Jour. of Math.*, vol. 30 (1898), p. 157.
† Burnside, *Proc. of the London Math. Soc.*, vol. 34 (1902), p. 159.
‡ Netto, Vorlesungen über Algebra, 1900, p. 328.
§ *Liouville*, vol. 11 (1846), p. 408.

Frobenius has employed the terms *inner* and *outer* automorphisms * instead of Hölder's terms cogredient and contragredient isomorphisms. These new terms have the advantage of being shorter and perhaps more natural than the older ones. Moreover, it seems desirable to have a term which explicitly states that a simple isomorphism is under consideration. This may be done by either of the new terms holomorphism or automorphism, which were introduced about the same time. Possibly it would be well to restrict the latter, as Frobenius has done, to express a simple isomorphism of a group with itself, while the former might express a general simple isomorphism.

It may be observed that the nomenclature here suggested is not free from objections. It is sometimes desirable to speak of a general isomorphism of a group with itself, and it would not appear unnatural to use the term automorphism for this purpose instead of restricting its use to a simple isomorphism of a group with itself. Moreover, Ritter, at the suggestion of Klein, has employed holomorphism for a general isomorphism and has used the term isomorphism to represent only a simple isomorphism. Burkhardt has followed this terminology in his Encyklopädie article.† It is open to the objection that the term isomorphism has been extensively used in a broader sense.

As the elements of a substitution group are not, in general, numbers, their laws of combination cannot be directly represented by relations between numbers. In a series of articles published in the *Mathematische Annalen* and in *Crelle*, Hoyer has developed a new theory for the algebraic solution of the problems of substitution groups.‡ These developments not only concern themselves with abstract group properties, such as the smallest number of generational relations and the finiteness of the order, but also with those relating to the representation as substitution groups, such as the degree of transitivity, the class, and the systems of imprimitivity. The general question of the determination of all the possible substitution groups of degree n is reduced to an algebraic one. Most of the developments are quite general and the results do not appear directly useful in advancing the theory along the older lines.

* Frobenius, *Berliner Sitzungsberichte*, 1901, p. 1324.

† Burkhardt, Encyklopädie der Mathematischen Wissenschaften, vol. 1, p. 220.

‡ Hoyer, *Math. Annalen*, vol. 49 (1897), p. 39; vol. 50 (1898), p. 499; vol. 51 (1899), p. 445; vol. 52 (1899), p. 550; *Crelle*, vol. 124 (1901), p. 102.

§ 4. Group Characteristics.

The independent generators of any abelian group A may be represented by cyclic substitutions in distinct sets of letters. If the orders of these cyclic substitutions are $\rho_1, \rho_2, \cdots, \rho_m$ the order of A is $\rho_1\rho_2\cdots\rho_m$. The group generated by any one of these substitutions of order ρ_a is simply isomorphic with the ρ_a roots of unity. If the roots which may be associated with the different generating substitutions are regarded as independent numbers just as the substitutions are independent, these roots will generate a group which is simply isomorphic with A. The numbers which constitute the operators of this group are the *group characteristics* of A.

A more abstract definition of the characteristics of A is as follows:* With every operator s of A there is associated a number which may be regarded as a function of s and is denoted by $x(s)$. This number must satisfy the following conditions: (1) It does not vanish for any value of s, (2) the equation $x(ss') = x(s)x(s')$ is satisfied for every pair of operators in A. Two such numbers as x, x' are to be regarded as different when there is some operator s_1 in A such that $x(s_1) \neq x'(s_1)$. The numbers thus defined are the group characteristics of A and it is not difficult to deduce the properties mentioned in the preceding paragraph from these more abstract definitions. The characteristics of an abelian group were already employed by Lagrange and Dirichlet † and their abstract definition is due to Dedekind.

Since 1896 Frobenius has published a series of memoirs in which he has developed a theory of group characteristics for non-abelian groups, which reduces to the preceding when the group is abelian.‡ He proves that the number of distinct characteristics is equal to the number of complete sets of conjugate operators of a group and that a characteristic of a given operator is the same as that of each of its conjugates. Moreover, each of the characteristics is the sum of roots of unity and the product of two characteristics can be represented as a linear function of all the characteristics of the group.

Weber gives a brief exposition of the theory of characteristics in the second edition of his Algebra, volume 2, 1899. More

* Cf. Weber, Lehrbuch der Algebra, vol. 2 (1899), p. 49.

† Frobenius, *Berliner Sitzungsberichte*, 1896, p. 985.

‡ *Loc. cit.*, 1896, pp. 985, 1343; 1897, p. 994; 1898, p. 501; 1899, pp. 330 and 482; 1900, pp. 303 and 516.

recently, Burnside has developed the same subject from a somewhat different standpoint in an article published in the *Proceedings of the London Mathematical Society.** In two subsequent articles published in the same volume he has applied the theory in the study of some properties of groups of odd order. In the first of these papers he proves that a linear substitution of odd order is reducible whenever all the coefficients are real and that there is no simple group of an odd composite order and of a prime degree. The main result arrived at in the second of these papers is that if an odd prime p is the smallest factor of the order of a group there must be an invariant subgroup of index p unless either p^4 or p^3q, where q is a prime factor of p^2+p+1, is a factor of the order.

To facilitate the application of characteristics Frobenius developed two methods which are more convenient in many special cases than the general method developed in 1896. The first of these is based upon the relations which exist between the characteristics of a group and those of a subgroup.† The case when this subgroup is invariant leads to especially interesting results. Moreover, a characteristic can be directly obtained by this method in case the group is represented as a doubly transitive substitution group. The second method is based upon the composition of the characteristics and is illustrated by means of the alternating and the symmetric groups of degree 6 as well as by the tetrahedral, octahedral and icosahedral rotation groups.‡

The *Berliner Sitzungsberichte* for 1901 contains three papers by Frobenius on solvable groups. In the first two of these group characteristics are extensively used. The main theorem of the first paper is as follows: If a group of order fg, f and g being relatively prime, contains a subgroup of order f composed of invariant operators then it must be the direct product of this subgroup and a subgroup of order g. This theorem was afterwards proved without the use of group characteristics by de Seguier in *Comptes Rendus*, volume 134 (1902). The second of these papers contains, among many others, the interesting theorem mentioned in the preceding paragraph that a group of class $n-1$ always contains a regular group of order n.

The principal theorem which is proved in the last of these three articles may be stated as follows: If p^λ is the highest power

* Vol. 33 (1901), p. 146.
† Frobenius, *Berliner Sitzungsberichte*, 1898, p. 501.
‡ *Loc. cit.*, 1899, p. 330.

of p which divides the order of a group H, and if no operator of H whose order is prime to p transforms a subgroup of order p^α into itself without being commutative with each one of its operators, then must H contain a subgroup of index p^λ which is composed of all the operators of H whose orders are not divisible by p. Frobenius observes that it is possible to state this theorem somewhat more generally; but in this case the statement becomes still more complex and we shall not present it here.

STANFORD UNIVERSITY,
June, 1902.

SHORTER NOTICES.

Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance. By M. CURTZE. Erster Theil. *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XII. Heft. Leipzig, Teubner, 1902. 336 pp. 16 Marks.

IT is a compliment to the monumental work of Professor M. Cantor that the activity in the field of the history of mathematics for the past twenty years has been almost entirely directed by him. The sole effort has been to supplement his work, to enter some of the innumerable doors which he has opened, to decipher the inscriptions upon the monument which he has erected. Hardly an article appears in the *Bibliotheca Mathematica*, relating to the period preceding the middle of the eighteenth century, that does not refer in some way to Cantor's work, and the Abhandlungen have been more or less under his direction for a quarter of a century.

Herr Curtze's latest contribution is an evidence in point, not merely in being dedicated to Professor Cantor on the occasion of his doctor's jubilee, but in that it elaborates certain details of his History for which elaboration scholars have been waiting.

Half of the work is given to the *Liber embadorum* of Abraham Savasorda (Sahib al Schorta, chief of the guards) as translated from the Hebrew by Plato of Tivoli in 1116, a treatise merely mentioned by Cantor.* The work has already been noticed by Curtze † as being one of the chief sources of

* Vorlesungen, vol. 2, p. 853.
† *Bibliotheca Mathematica*, vol. 1, 3d series (1900), p. 501.

Leonardo of Pisa's Practica Geometriæ, and Braunmühl has testified * to its importance in the history of trigonometry. Of course it has also been more or less known to other historians, particularly to Steinschneider † who has devoted so much attention to Jewish mathematics.‡ The work itself, however, has been closed to the world at large until the appearance of the present edition.

It speaks well for the brotherhood of nations, as also of scholars, that the French government should permit the two manuscripts, which form the basis for the present edition, to be sent from the National Library and be placed at the disposal of Herr Curtze. As a result we have, on opposite pages, the Latin and German texts, the former with the variations in reading of the two manuscripts.

The most interesting fact relating to the treatise is that it shows at the same time the chief source of Fibonacci's work on geometry, and the originality of the latter. Both of these facts appear throughout the treatise, and are emphasized in the footnotes. While the work is devoted chiefly to geometry, it is also valuable as giving insight into the Spanish arithmetic of the period. It shows that, among the learned class, the subject was of the Boethian stamp, the Hindu numerals being unknown, and the definitions being mainly the conventional ones of the middle ages. *Mutabemini numeri* and *almugesem numeri* are new names, however, and the explanation of the meaning of the product of a line by a line shows a broadening of definition due to the Arab influence. Abraham solves certain equations, as $x^2 + b = ax$ (giving the double root), $x^2 + y^2 = a^2$, $x \pm y = b$, and $x^2 + y^2 = a^2$, $xy = b$, of course geometrically. The geometry, as was the custom of the time, due both to Arab and to Roman influence, is largely given to mensuration. The value of π is given as $3\frac{1}{7}$ or 377/120 (that of Ptolemy), and the treatment of the circle is followed by some discussion of trigonometry, the table of chords being probably the oldest that is known in Latin form.

The second part of the volume is devoted to the correspondence of Regiomontanus with Giovanni Bianchini, Jacob von Speier and Christian Roder. This correspondence is preserved in the city library at Nürnberg, and although once published,§

* Geschichte der Trigonometrie, vol. 1, p. 93.

† *Zeitschrift für Mathematik und Physik*, XII, 1.

‡ See also his articles in the *Bibliotheca Mathematica*.

§ Christophori Theophili de Murr Memorabilia Bibliothecarum Publicarum Norimbergensium et Universitatis Altorfinæ. Pars 1., M.DCC.LXXXVI.

its reappearance is valuable both because it corrects certain errors in Murr's transcription, and because it now becomes generally accessible.

Regiomontanus was the leader of his generation in astronomy and mathematics, and his correspondence with Bianchini, who was court astronomer to the Duke of Ferrara, Speier, who was court astrologer to the Prince of Urbino, and Roder, the professor of mathematics at the University of Erfurt, throws much light upon the practical astronomical work of the fifteenth century. The correspondence is in Latin and no translation is given.

Altogether, this number of the Abhandlungen is one of the most valuable that have appeared, and the tendency to publish the sources for the history of mathematics is one that will meet the hearty commendation of scholars.

DAVID EUGENE SMITH.

Gauss' wissenschaftliches Tagebuch, 1796–1814. Mit Anmerkungen herausgegeben von FELIX KLEIN. Reprinted from the *Festschrift zur Feier des hundertfünfzigjährigen Bestehens der Königlichen Gesellschaft der Wissenschaften zu Göttingen*. Berlin, Weidmannsche Buchhandlung, 1901, 8vo., 44 pp.

As a youth not quite nineteen years old Gauss began jotting down in a copy-book memoranda, always, unfortunately, of the very briefest sort, of the great mathematical discoveries he was making. The entries in this Scientific Diary (*Catalogus*, Gauss calls it) are in Latin, and begin with a statement dated March 30, 1796, to the effect that Gauss had found a construction for the regular polygon of seventeen sides. From this date the entries follow each other in rapid succession, there being no less than 112 in the next four years and a quarter. From here on they become more irregular, and there are only 34 entries during the following fourteen years. Such a diary as this, written by any great mathematician, would be of the greatest interest, as illustrating, even with all its gaps and obscurities, the order in which the mathematical ideas developed in his mind and the form they first took; but there is probably no mathematician in whose case it could be even approximately as valuable as in the case of Gauss. For it is well known that ideas, many of them of the first importance, poured in on Gauss's mind in his early youth in such numbers that, as he himself

said, he was at times hardly able to master them; and many of these ideas, especially those concerning elliptic functions, were never incorporated in memoirs.

This diary has now been given to the public by Professor Klein, who has enhanced its value very greatly by attaching to a large number of its entries explanatory notes pointing out their relation to other published or unpublished writings of Gauss. Professor Klein explains, however, that this publication is to be regarded as a preliminary one only, since the diary is to be printed, with a more extended commentary, in Volume X, of Gauss's Collected Works.

A portrait of Gauss at the age of twenty-six, which has never before been published, serves as frontispiece, and one of the pages of the diary is reproduced in facsimile.

MAXIME BÔCHER.

Zur Integration partieller Differentialgleichungen. DR. KARL BOEHM. Leipzig, Teubner, 1900. 8vo. 55 pp. Mk. 1.80.

THE above paper, printed in book form, deals exclusively with the formal solution of partial differential equations, and of systems of such equations, by means of series, the convergence of which is not investigated.

The author starts with a given equation, or set of equations, of any order, forms their successive derived equations, and counts the number of derivatives of highest order compared with the number of equations, at each stage. He then shows, for a single equation, that the given equation can be solved with respect to any exceptional * derivative, and finally that a power series expansion may be obtained for the solution, which will formally satisfy the given equation, when arbitrary constant values have been preassigned for certain suitable derivates; all this at a point where the equation is not "singular" with the respect to the exceptional derivative chosen. Similar results are obtained for a system of equations, under certain restrictions.

The work doubtless has real merit, but the reviewer does not feel justified in entering further into detail on account of the failure to discuss the convergence of the series in question: an omission which the author acknowledges in several places, and which renders the importance of the paper rather doubt-

* "Ausgezeichnet;" the definition is too intricate to repeat here.

ful. For it reduces the work to a questionable revision, not of Weierstrass and Kowalewsky, but of Cauchy, as the author confesses in his preface; certainly in fact as well as in principle not a modern tendency. In fact the author's only plea for a very complicated piece of reasoning, which might indeed have been put into far simpler form, is that "it gives results even for certain singular points, and fails only for the points which we have called *wholly singular*"— for several reasons a doubtful advantage, especially in view of the unfinished state of the work.

In conclusion, the material of the book would seem to be more fitted for a thesis; or for publication in a journal (in which case nearly twenty pages of repetition and unnecessary examples might profitably have been omitted); or for preservation unpublished, awaiting some of the proofs of convergence of which the author is happily sanguine. In its present form it may serve, if at all, perhaps to some student who will furnish the major portion of the whole by giving these convergence proofs. It is not in any sense a text-book or a treatise, and it is certainly not a book for general purchase.

E. R. HEDRICK.

The Measurement of General Exchange Value. By CORREA MOYLAN WALSH. New York, The Macmillan Company, 1901. 8vo., xiv + 580 pp. $3.00.

THIS is the most exhaustive work ever brought out on the theory of index numbers, embodies a vast deal of labor and acute logic, and will be a mine of information to future investigators. In so well worked ground there is necessarily much that is familiar to students of the subject, though hitherto inaccessible outside the largest libraries. But this editorial portion of the material — the most excellent bibliography, the index, the symbolized summaries of the methods of his predecessors — may prove to be the most valuable part of Mr. Walsh's work.

It is not in place in this BULLETIN to enter on the purely economic aspect of the subject, which is covered by an extended notice by Professor F. Y. Edgeworth in the *British Economic Journal* for September, 1901. But the mathematician and the practical statistician may find themselves puzzled by the confident rejection of the aid of the calculus of probabilities, elsewhere found so helpful both to theory and practice in dealing with complex problems of mensuration. Doubtless—as Professor

Edgeworth points out—the variations between two given series of index numbers according to different systems are within the range of probable error, and therefore inconclusive. What an amount of controversy might have been saved by general recognition of this fact.

To those interested in the theory of averages mention may be made of the appendices, in which are detailed the elementary propositions relating to the arithmetic, geometric and harmonic means, with single and multiple weighting.

The logical method of the work is admirable, its index most complete, but it leaves the reader with the impression that further work must be done and that no one solution will cover what is really a considerable number of independent problems.

J. M. GAINES.

NOTES.

THE closing (October) number of volume 3 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the groups of order p^m which contain operators of order p^{m-2}," by G. A. MILLER; "On the circuits of plane curves," by C. A. SCOTT; "Note on the real inflexions of plane curves," by C. A. SCOTT; "La théorie des plaques élastiques planes," by J. HADAMARD; "Covariants of systems of linear differential equations and applications to the theory of ruled surfaces," by E. J. WILCZYNSKI; "On the rank, order and class of algebraic minimum curves," by A. S. GALE; "On superosculating quadric surfaces," by H. MASCHKE; "Algebraic transformations of a complex variable realized by linkages," by A. EMCH; "On the determination of the distance between two points in space of m dimensions," by H. F. BLICHFELDT; "A definition of abstract groups," by E. H. MOORE; notes and errata: volumes 1, 2, 3.

THE October number (volume 24, number 4) of the *American Journal of Mathematics* contains: "On systems of linear differential equations of the first order," by M. BÔCHER; "On the quaternary linear homogeneous group and the ternary linear fractional group," by T. M. PUTNAM; "On cardinal numbers," by A. N. WHITEHEAD; "On a method of constructing all the groups of order p^m," by G. A. MILLER; "Non-euclidean

properties of plane cubics and of their first and second polars," by H. F. STECKER.

AT the meeting of the British association for the advancement of science held at Belfast, September 10–12, the president of Section A, Professor JOHN PURSER, took as the subject of his opening address an historical sketch of the Irish school of mathematics and physics. Sir NORMAN LOCKYER was elected president of the association for the coming year. The meeting in 1903 will be held at Southport, beginning September 9, and in 1904 at Cambridge. It is probable that the meeting in 1905 will be held at Cape Town.

THE several universities below offer during the winter semester of the current academic year courses in mathematics as follows:

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Differential calculus and the elements of integral calculus, four hours; Introduction to the theory of functions, two hours; Exercises in the elements of higher mathematics (algebra, analytic geometry, differential and integral calculus), two hours; Exercises in the mathematical seminar, one hour.—By Professor E. NETTO: Definite integrals and their applications, four hours; Analytic geometry of space, four hours; Exercises in the mathematical seminar.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Algebra, four hours; Theory of plane algebraic curves, two hours; Mathematical seminar, two hours.—By Professor E. STUDY: Non-euclidean geometry, two hours; Mechanics, four hours; Mathematical seminar, two hours.—By Professor G. KOWALEWSKI: Introduction to analytic geometry, two hours; Theory of continuous transformation groups, two hours; On Fourier's series (with applications to mathematical physics), two hours.

UNIVERSITY OF HALLE.—By Professor A. WANGERIN: Integral calculus, with exercises, five hours; Calculus of variations, two hours; Hydrodynamics, two hours; Exercises in the mathematical seminar, one hour.—By Dr. H. GRASSMANN: Exercises in descriptive geometry, one hour; Analytic geometry of space, two hours; Elements of descriptive geometry, two hours.

UNIVERSITY OF KÖNIGSBERG.—By Professor F. MEYER: Exercises in integral calculus, one hour; Integral calculus, four hours; Differential geometry, four hours; Exercises in differential geometry, one hour.—By Professor A. SCHOENFLIES: Mathematical seminar, two hours; Applications of elliptic functions to geometry, mechanics and mathematical physics, four hours.—By Professor L. SAALSCHÜTZ: Exercises in algebraic analysis, one hour; Theory of Fourier's series, two hours; Introduction to algebraic analysis, two hours.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY: Algebraic analysis, four hours; Mathematical seminar, two hours.—By Professor A. E. HESS: Integral calculus, five hours; Spherical trigonometry and its applications, two hours; Mathematical seminar, two hours.—By Dr. F. VON DALWIGK: Analytic geometry of space, especially surfaces of the second degree, four hours; Analytic geometry and graphic statics, with exercises, two hours; Advanced analytic geometry, one hour.—By Dr. H. JUNG: Algebra, four hours; Differential and integral calculus, four hours.

UNIVERSITY OF MÜNSTER.—By Professor W. KILLING: Analytic mechanics, five hours; Analytic geometry, five hours; Theory of transformation groups, two hours; Exercises in the mathematical proseminar, two hours; Exercises in analytic geometry, one hour.—By Professor R. VON LILIENTHAL: Differential and integral calculus, five hours; Introduction to the theory of differential equations, four hours; Exercises in the mathematical seminar, one hour.—By Dr. M. DEHN: Elliptic functions, three hours; Graphic statics, two hours; Exercises in the theory of functions, one hour.

UNIVERSITY OF INNSBRUCK.—By Professor O. STOLZ: Theory of double integrals, two hours; Elements of the calculus of variations, one hour; Exercises in these subjects in the mathematical seminar, one hour; Theory of functions of complex variables according to Cauchy and Weierstrass, three hours.—By Professor W. WIRTINGER: Higher algebra, three hours; Abelian functions, two hours; Mathematical seminar, two hours.—By Professor K. ZINDLER: Analytic geometry of the plane and of space, five hours; Mathemetical seminar, one hour.

UNIVERSITY OF NEUCHÂTEL.—By Professor L. ISELY: Infinitesimal calculus, including differential equations, three

hours; Analytic geometry of space, two hours; Theory of numbers, one hour.—By Professor L. GABEREL: General theory of functions, two hours; Abelian functions, one hour.

UNIVERSITY OF PRAGUE.—By Professor G. PICK: Differential and integral calculus, five hours; Mathematical seminar, two hours.—By Professor J. A. GMEINER: Analytic geometry, three hours; Double integrals, two hours; Exercises in analytic geometry, one hour.—By Dr. W. WEISS: Elements of descriptive geometry, two hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. VON BRILL: Introduction to higher mathematics, four hours; Non-rigid systems and the mechanics of Hertz, three hours; Seminar, two hours.—By Professor H. STAHL: Higher algebra, three hours; Applications of the theory of functions, four hours; Seminar, two hours. — By Professor L. MAURER: Definite integrals, two hours; Differential equations, two hours; The theory of ternary forms, one hour, with exercises, one hour; Exercises in higher analysis, two hours.

UNIVERSITY OF VIENNA.—By Professor G. R. VON ESCHERICH: Elements of the differential and integral calculus, with particular reference to their common applications, five hours; Exercises relating to these lectures, one hour; Proseminar, one hour; Seminar, two hours.—By Professor L. GEGENBAUER: Integral calculus and calculus of variations, three hours; Theory of spherical and cylindrical functions, with applications to problems of theoretical physics, two hours; Exercises in the mathematical proseminar, one hour; Exercises in the mathematical seminar, two hours.—By Professor F. MERTENS: Theory of numbers, five hours; Exercises in the mathematical seminar, two hours; Exercises in the mathematical proseminar, one hour.—By Professor G. KOHN: Introduction to synthetic geometry, four hours; Exercises relating to these lectures, one hour; Theory of invariants with applications to geometry, two hours.—By Dr. A. TAUBER: The mathematics of life insurance, four hours; Exercises relating to this course, two hours. —By Dr. E. BLASCHKE: Introduction to mathematical statics, second course, three hours.—By Dr. R. DAUBLEBSKY VON STERNECK: Applications of differential and integral calculus to geometry, two hours; Theory of numbers, one hour.—By Dr. K. CARDA: Introduction to the theory of contact transformations.—By Dr. R. SCHRAM: Method of least squares, one

hour.—By Professor H. HARTL: Elements of descriptive geometry, with applications in constructions, four hours; Geodetic coördinates, with computations, one and one half hours.

PROFESSOR F. SCHOTTKY, of Marburg, has been appointed to succeed the late Professor L. FUCHS at the University of Berlin.

DR. ALFRED LOEWY, of the University of Freiburg, has been promoted to an associate professorship of mathematics.

DR. H. HOHENNER of Munich has been appointed professor of geodesy in the Technical high school in Stuttgart.

PROFESSOR D. A. GRAVÉ of Kharkov has been called to a professorship of mathematics in the University of Kieff.

DR. E. HAENTZSCHEL has been appointed professor of mathematics in the Technical high school in Berlin.

PROFESSORS R. HAUSSNER, of Giessen, and A. KRAZER, of Strassburg, have been called to the Polytechnic school at Carlsruhe.

PROFESSOR L. M. DEFOE, of the department of mathematics of the Missouri State University, has been appointed professor in charge of the department of mechanics in that institution. Professor Defoe will enter upon his new duties on his return from Europe where he is spending a year on leave of absence.

DR. F. H. SAFFORD has been appointed instructor in mathematics at the University of Pennsylvania.

PROFESSOR W. H. METZLER, of Syracuse University, has been elected to membership in the Royal Society of Edinburgh.

AT Yale University, Mr. W. D. A. WESTFALL has been appointed instructor in mathematics in the academic department, and Mr. BURKE SMITH in the Sheffield scientific school. Dr. H. E. HAWKES has resumed his duties as instructor. Dr. E. B. WILSON is spending a year's leave of absence abroad.

JOHN WHELDON & Co., 36 Great Queen Street, London, W. C., have recently issued a catalogue (No. 14) of second-hand mathematical works, containing nearly 800 entries.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AHRENS (W.). See ENCYKLOPÄDIE.

BACHMANN (P.). Niedere Zahlentheorie. (In 2 Teilen.) Teil I. Leipzig, Teubner, 1902. 8vo. 10 + 402 pp. Cloth. (Teubner's Sammlung mathematischer Lehrbücher, Vol. X, 1.) M. 14.00

BOPP (K.). Antoine Arnauld, der grosse Arnauld, als Mathematiker. (Diss.) Heidelberg, 1902. 8vo. 49 pp.

CURTZE (M.). Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance. (In 2 Teilen.) Teil 2. Leipzig, Teubner, 1902. 8vo. Pp. 337–627. M. 14.00

DESSENON (E.). Eléments de géométrie analytique, à l'usage des candidats à l'École centrale et des élèves de première année de la classe de mathématiques spéciales. 3e édition. Paris, Hachette, 1902. 8vo. 523 pp. (Cours complet d'études scientifiques.) Fr. 7.50

DOEHLEMANN (K.). Geometrische Transformationen. Teil 1: Die projektiven Transformationen nebst ihren Anwendungen. Leipzig, Göschen, 1902. 8vo. 7 + 322 pp. Cloth. (Sammlung Schubert, No. XXVII.) M. 10.00

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. (In 7 Bänden.) Vol. I: Arithmetik und Algebra, redigiert von W. F. Meyer. Heft 7: R. Mehmke, Numerisches Rechnen; W. Ahrens, Mathematische Spiele; V. Pareto, Anwendungen der Mathematik auf Nationalökonomie. Leipzig, Teubner, 1902. 8vo. Pp. 993–1120.

EYSANK (J. VON). Einige Aufgaben aus der analytischen Geometrie. (Progr.) Wien, 1902. 8vo. 25 pp.

FOUËT (E. A.). Leçons élémentaires de la théorie des fonctions analytiques. Partie I. Paris, 1902. 8vo. Pp. 1–310. Fr. 7.50

FRICKE (R.). Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. Teil 1. 2ter Abdruck. Braunschweig, Vieweg, 1902. 8vo. 9 + 80 pp. M. 2.00

FUCHS (R.). Ueber lineare homogene Differentialgleichungen, deren Substitutionsgruppe von einem in den Koeffizienten auftretenden Parameter unabhängig ist. (Progr.) Berlin, 1902. 4to. 23 pp.

FUHRMANN (W.). Kollineare und orthologische Dreiecke. (Progr.) Königsberg, 1902. 4to. 20 pp.

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie, mit zahlreichen Uebungsbeispielen. Teil II: Die analytische Geometrie des Raumes, von F. Rudio. 3te, verbesserte Auflage. Leipzig, 1901. 8vo. 10 + 186 pp. Cloth. M. 3.00

GROAT (B. F.). An introduction to the summation of differences of a function. An elementary exposition of the nature of the algebraic processes replaced by the abbreviations of the infinitesimal calculus. 4 + 43 pp. Seven lessons in theory of inversions of order and determinants. 32 pp. Minneapolis, Wilson, 1902. 8vo. Cloth.

GRÜNWALD (A.). Geodätische Linien auf dem Ellipsoide. (Progr.) 1902. 8vo. 27 pp., 1 plate.

HADAMARD (J.). La série de Taylor et son prolongement analytique. [Paris] Naud, 1901. 12mo. 10 + 102 pp. Cloth. (*Scientia*, série physico-mathématique, No. 12.) Fr. 2.00

HALSTED (G. B.). The foundations of geometry. 8vo. (*The Open Court* 16, pp. 513–521.)

HAUTH (R.). Ueber die Flächen, von deren Krümmungslinien ein System in parallelen Ebenen sich befindet. (Progr.) Metten, 1902. 8vo. 33 pp.

HOYER. Andreas Gärtner, der "sächsische Archimedes." (Progr.) Dresden, 1902. 4to. 21 pp., 1 plate.

KADESCH (A.). Ueber die Einhüllungsflächen von Potenzflächenscharen. (Progr.) Wiesbaden, 1902. 4to. 45 pp.

KINKELIN (H.). Quadraturen. (Progr.) Basel, Schwabe, 1902. 4to. 30 pp. M. 1.60

KOCH (W.). Die Eigenschaften der Kurven 4ten Grades mit 2 Doppelpunkten, hergeleitet mittelst elliptischer Funktionen. (Progr.) Züllichau, 1902. 4to. 14 pp.

LAURENT (H.). Sur les principes fondamentaux de la théorie des nombres et de la géométrie. [Paris] Naud, 1902. 12mo. 68 pp. Cloth. (*Scientia*, série physico-mathématique, No. 20.) Fr. 2.00

LODGE (A.). Differential calculus for beginners. With an introduction by Sir O. J. Lodge. London, Bell, 1902. 12mo. 304 pp. Cloth. 4s. 6d.

MAUPIN (G.). Opinions et curiosités touchant la mathématique. 2e série. Paris, Naud, 1902. 8vo. 338 pp.

MEHMKE (R.), MEYER (W. F.). See ENCYKLOPÄDIE.

MIORINI (W. VON). Ueber eine Erweiterung der Sätze von Pascal und Brianchon. (Progr.) Wien, 1902. 8vo. 12 pp., 1 plate.

MORITZ (R. E.). Generalization of the differentiation process. 4to. (*American Journal of Mathematics* 24, pp. 257–301.)

PARETO (V.). See ENCYKLOPÄDIE.

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte I: Calcolo differenziale. 2a edizione completamente riveduta. Milano, Hoepli, 1902. 16mo. 12 + 331 pp. (Manuali Hoepli.)

PILGRAM (A.). Die Scharschar der Kegelschnitte, die ein gegebenes Tangentendreieck haben. (Diss.) Erlangen, 1902. 8vo. 27 pp., 1 plate.

RUDIO (F.). See GANTER (H.).

SCHUBERT (H.). Niedere Analysis. Teil I: Kombinatorik, Wahrscheinlichkeitsrechnung, Kettenbrüche und diophantische Gleichungen. Leipzig, Göschen, 1902. 8vo. 7 + 181 pp. Cloth. (Sammlung Schubert, No. V.) M. 3.60

VERONESE (G.) Les postulats de la géométrie dans l'enseignement. 8vo. (Deuxième Congrès international des mathématiciens, Paris, 1900, pp. 433–450.)

VILLANI (N.). Ricerche matematiche sulle misure antiche e il sistema antico delle misure romane. Lanciano, Carabba, 1902. 16mo. 7 + 66 pp. Fr. 1.00

ZINDLER (K.). Liniengeometrie mit Anwendungen. Vol. I. Leipzig, Göschen, 1902. 8vo. 8 + 380 pp. Cloth. (Sammlung Schubert, No. XXXIV.) M. 12.00

ZÜHLKE (P.). Ueber die geodätischen Linien und Dreiecke auf den Flächen konstanten Krümmungsmasses und ihre Beziehungen zur sogenannten nicht-Euklidischen Geometrie. (Progr.) Charlottenburg, 1902. 8vo. 36 pp., 1 plate.

II. ELEMENTARY MATHEMATICS.

ASCARZA (V. J.) y SOLANA (E.). Colección de problemas de aritmética razonados y resueltos analíticamente; aprobado de real orden para servir de texto. 3a edición corregida y aumentada. Madrid, Imp. Moderna, 1902. 8vo. 183 pp. Fr. 1.50

BROOKS (E.). Plane and solid geometry; a complete course in the elements of the science. Revised edition. Philadelphia, Christopher Sower Co. [1902]. 12mo. 415 pp. Cloth. $1.25

CHAILAN (E.). Géométrie, à l'usage des élèves des classes de lettres, conforme au programme de la première partie du baccalauréat de l'enseignement secondaire classique. 2e édition. Paris, Poussielgue, 1902. 16mo. 8 + 240 pp. (Alliance des maisons d'éducation chrétienne.)

CHANCELLOR (W. E.). Elementary school mathematics by grades. Seventh book: arithmetic, geometry, and algebra. New York, Globe School Book Co. [1902]. 12mo. 176 pp. Cloth. (Globe series.) $0.28

DAVISON (C.). Easy mathematical problem papers; with answers. London, Blackie, 1902. 12mo. 12 pp. Cloth. 2s. 6d.

FLETCHER (W. C.). Elementary geometry. London, Arnold, 1902. 12mo. 84 pp. Cloth. 1s. 6d.

FÜCHTJOHANN (H.). Lösung der Aufgaben in J. R. Boyman's Lehrbuch der Planimetrie. 2ter Teil: Aufgabe 734–1244. Bonn, Cohen, 1902. 8vo. 212 pp. M. 4.20

GIKA (D.) and MUROMTSEV (A.). Geometrical problems. Part II: Solid geometry. Moscow, 1902. 8vo. 163 pp. (Russian.) M. 3.00

GODT (W.). Ueber einige sogenannte merkwürdige Punkte des Dreiecks. I. (Progr.) Lübeck, 1902. 4to. 23 pp.

HOPKINS (G. I.). Inductive plane geometry; with numerous exercises, theorems, and problems for advance work. New revised edition. Boston, Heath, 1902. 12mo. 6 + 208 pp. Hf. leather. $0.75

JELINEK (L.). Mathematische Tafeln für technische Lehranstalten, besonders für höhere Gewerbeschulen. 3te Auflage; nach der neuen Rechtschreibung berichtigter, sonst im wesentlichen unveränderter Abdruck der 2ten Auflage. Wien, Pichler, 1902. 8vo. 223 pp. Cloth. M. 2.00

LORENZ (K.). Das Rechnen mit unvollständigen Dezimalbrüchen. (Progr.) Weidenhofen, 1902. 8vo. 22 pp.

LÖSUNGEN der Absolutorial-Aufgaben aus der Mathematik an den humanistischen Gymnasien Bayerns. 8ter Nachtrag: 1902. München, Pohl, 1902. 8vo. 7 pp. M. 0.30

MAHLER (G.). Ebene Geometrie. 3te, verbesserte Auflage, 2ter Abdruck. Leipzig, Göschen, 1902. 12mo. 158 pp. Cloth. (Sammlung Göschen, No. 41.) M. 0.80

MUROMTSEV (A.). See GIKA (D.).

PETERS (F.). Ziegler's graphische Darstellung der trigonometrischen Funktionen nebst Tafeln zur Konstruktion bestimmter Winkel und Linien. Ein praktisches Hilfsmittel beim geometrischen Zeichnen. Wiesbaden, Kreidel, 1902. 8vo. 22 pp., 6 plates. Cloth. M. 3.00

SÁNCHEZ CASADO (F.) y SÁNCHEZ RUEDA (E.). Prontuario de geometría y trigonometría, para uso de los alumnos de los institutos, colegios y seminarios. 7a edición corregida y notablemente aumentada por E. Sánchez Rueda. Madrid, Hernando y de Jubera, 1902. 8vo. 127 pp. Fr. 1.00

SOLANA (E.). See ASCARZA (V. J.).

III. APPLIED MATHEMATICS.

APARICI (R.). Lecciones de geometría descriptiva. 2a edición. (Rectas y planos, poliedros, superficies cilíndricas, cónicas y de revolución; planos acotados.) Madrid, Lib. Gutenberg de José Ruiz, 1903. 4to. 8 + 32 + 8 pp. Fr. 8.00

APPELL (P.). Traité de mécanique rationnelle (cours de mécanique de la Faculté des sciences de Paris). Vol. III: Equilibre et mouvement des milieux continus. 3e fascicule. Paris, Gauthier-Villars, 1903. 8vo. Pp. 457–558.

ATKINSON (A. A.). Electrical and magnetic calculations. New York, 1902. 8vo. 7 + 310 pp. Cloth. $1.50

BRÜSCH (W.). Grundriss der Elektrotechnik für technische Lehranstalten. Leipzig, Teubner, 1902. 8vo. 11 + 168 pp. Cloth. M. 3.00

BURZHEIEV (S.). The elements of the resistance of materials and graphostatics, with applications to the construction of machines. St. Petersburg, 1902. 8vo. 207 pp. (Russian.) R. 1.50

CESÀRO (E.). Sur un problème de propagation de la chaleur. 8vo. (*Bulletin de l'Académie royale de Belgique*, classe des sciences, 1902, pp. 387–404.)

CIURLO (M.). Sul principio dei lavori virtuali; memoria. Genova, Istituto Sordomuti, 1902. 8vo. 9 pp.

FITZGERALD (G. F.). Scientific writings, collected and edited, with a historical introduction by J. Larmor. London, Longmans, 1902. 8vo. Portrait. Cloth. 15s.

GENOVINO (G.). La variabilità degli angoli che la congiungente le punte del crescente lunare forma col circolo di declinazione e col verticale del centro della luna e problemi relativi. Pistoia, Niccolai, 1902. 8vo. 36 pp.

——. L'epoca della massima visibilità del planeta Venere. Firenze, Landi, 1902. 8vo. 13 pp.

——. Le epoche in cui la luna in una sua rivoluzione intorno alla terra si trova: 1° alla minima distanza da due stelle; 2° ad ugual distanza da due stelle; 3° sul circolo massimo che passa per due stelle. Pistoia, Niccolai, 1902. 8vo. 13 pp.

HARTMANN (R.). Beiträge zur Wirbelbewegung. (Diss.) Braunschweig, 1902. 8vo. 31 pp.

HAUG (J.). Ueber die Drehung eines starren Körpers um seinen Schwerpunkt. (Progr.) München, 1902. 8vo. 17 pp.

HEUN (K.). Formeln und Lehrsätze der allgemeinen Mechanik, in systematischer und geschichtlicher Entwickelung dargestellt. Leipzig, Göschen, 1902. 8vo. 8 + 112 pp. Cloth. M. 3.50

HOFF (J. H. VAN'T). Acht Vorträge über physikalische Chemie, gehalten auf Einladung der Universität Chicago. Braunschweig, Vieweg, 1902. 8vo. 7 + 81 pp. M. 2.50

JOHNSON (L. J.). The determination of unit stresses in the general case of flexure. 8vo. 39 pp. (*Journal of the Association of Engineering Societies* 27, 1892.)

KLOSS (M.). Analytisch-graphisches Verfahren zur Bestimmung der Durchbiegung zwei- und dreifach gestützter Träger. Mit besonderer Berücksichtigung von Drehstrommotorenwellen. (Diss.) Berlin, Polytechnische Buchhandlung, 1902. 8vo. 128 pp., 4 plates. M. 3.00

KOTTENBACH (R.). Zur didaktischen Behandlung einiger Fragen der Mechanik. (Progr.) Troppau, 1902. 8vo. 28 pp.

KRIEMLER (C. J.). Labile und stabile Gleichgewichtsfiguren vollkommen elastischer auf Biegung beanspruchter Stäbe mit besonderer Berücksichtigung der Knickvorgänge. (Habilitationsschrift.) Karlsruhe, Braun, 1902. 4to. 56 pp., 10 plates.

LANGER (K.). Direkte Konstruktion der Konturen von Rotationsflächen II. Ordnung in orthogonaler Darstellung. (Progr.) Horn, 1902. 8vo. 12 pp.

LARMOR (J.). See FITZGERALD (G. F.).

OVAZZA (E.). Urti ed esplosioni; lezioni di dinamica applicata. Torino, Lattes, 1902. 8vo. 155 pp. Fr. 5.00

PETRUS (A.). Beiträge zur Theorie der Herpolhodie Poinsots. (Diss.) Halle, 1902.

POEZL (W.). Anfangsgründe der darstellenden Geometrie, enthaltend die geradlinigen ebenen Gebilde, zum Schulgebrauche zusammengestellt. 21e Auflage. München, Ackermann, 1902. 8vo. 4 + 58 pp. M. 1.20

——. Elemente der darstellenden Geometrie, für höhere Lehranstalten zusammengestellt. 2 Teile. 2te Auflage. Teil I: Geradlinige Gebilde. 6 + 112 pp. Teil 2: Krummflächige Gebilde. 6 + 111 pp. München, Ackermann, 1902. 8vo. M. 4.00

QUESNEVILLE (G.). Nouvelle dioptrique des rayons visuels; théorie nouvelle de la lunette de Galilée. Paris, Hermann, 1902. 8vo. 64 pp.

RÜLF (B.). Der Reguliervorgang bei Dampfmaschinen. (Diss.) Berlin, 1902. 4to. 15 pp.

SANTEL (A.). Bemerkungen zur Didaktik einiger Kapitel der Mechanik. (Progr.) Görz, 1902. 8vo. 41 pp.

SCHEEFFER (E.). Ueber stabiles Schwimmen homogener Körper. (Progr.) Danzig, 1902. 4to. 52 pp.

SPIRGATIS (E.). Berechnung der Fahrzeiten aus den Zugkräften der Dampflokomotiven. Leipzig, Spirgatis, 1902. 8vo. 68 pp. M. 3.50

STOLLE (R.). See WEICKERT (A.).

TÄGERT (W.). Ueber Schwankungen der Drehungsachse der Erde im Innern des Erdkörpers. (Progr.) Siegen, 1902. 4to. 30 pp.

WEBER (R. H.). Elektromagnetische Schwingungen in Metallröhren. (Habilitationsschrift.) Heidelberg. 1902. 8vo. 33 pp.

WEICKERT (A.) und STOLLE (R.). Praktisches Maschinenrechnen. 5te Auflage (9tes und 10tes Tausend). Berlin, Polytechnische Buchhandlung, 1902. 8vo. 7 + 292 pp. Cloth. M. 5.00